This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

B

~~2.11.8~~

1076

B.e.15

6.11
QA
35
.M96

Probably from the Marqs of [illegible] library

see dedication ?

~~2.11.8~~

1076

B.e.15

— 6.11
QA
35
.M96

Probably from the Marquis of [illegible] library

see dedication

A

Mathematical Treatiſe:

Containing a SYSTEM of

CONIC-SECTIONS;

WITH the DOCTRINE of

FLUXIONS and FLUENTS,

APPLIED TO

VARIOUS SUBJECTS; *viz.*

To the finding the *Maximums* and *Minimums* of Quantities; Radii of Evolution, Refraction, Reflection; Superficial and Solid Contents of Curvilinear Figures; Rectification of Curve-lines; Centers of Gravity, Oſcillation and Percuſſion.

AS ALSO

To the Reſolution of a Select COLLECTION of the moſt uſeful, and many new, PHYSICO-MATHEMATICAL Problems.

By *JOHN MULLER.*

LONDON:
Printed by T. GARDNER, in *Bartholomew-Cloſe*, for the Author; and Sold by W. INNYS and R. MANBY, in St. *Paul's Church-Yard*, and J. NOURSE without *Temple-Bar*, M,DCC,XXXVI.

Hist. of Sci
Bowes
12-11-33
28413
6-27-34

TO

HIS GRACE,

JOHN,

Duke of *ARGYLL* and

GREENWICH.

May it pleaſe Your GRACE.

AS the ſeveral Improvements made in the Arts and Sciences, are chiefly owing to the Countenance, Influence, and Beneficence of the Great; ſo all Attempts to make any Progreſs in them, naturally ſeek Your Grace's Protection: For you have always ſhewn a noble Inclination to promote every kind of uſeful Knowledge, and polite Learning.

This, and Your Grace's known Goodneſs, are the Motives that encouraged me to hope for a favourable Reception of my Performance; to which the Patronage of your great Name, is an honour that requires all my Gratitude, and engages me ever to be, with a profound Reſpect and Veneration,

MY LORD,

May it pleaſe Your GRACE,

Your GRACE's *moſt obedient*
and moſt devoted
Servant,

John Muller.

From the Tower of
LONDON, *May*
16, 1736.

PREFACE:

THE excellency and uſefulneſs of Mathematics in general, and more eſpecially of thoſe branches which we conſider in this Treatiſe, have been ſo clearly ſhewn by many very able Authors, that I need not enlarge upon the ſame: I only intend by this Preface to give my Readers ſome of the Reaſons, which have induced me to write upon a Subject, which has been already very largely handled by ſome of the moſt conſiderable Mathematicians among the Moderns.

Notwithſtanding that a great part of this Work contains the ſame Truths, as have been demonſtrated by others; I conceived that the pains of learning them, and the time neceſſary to be ſpent in thoſe Inquiries, would be conſiderably leſſened, if by a proper Order or Method, the connection of the ſeveral Properties of the Conic-Sections and other Curves, was ſhewn in a clearer, and, at the ſame time, in a more conciſe manner than what is generally to be met with in Books written on that Subject. This Conciſeneſs and Natural Order is what I have aim'd at in this Treatiſe.

In the firſt place, I have gathered together and brought into a little compaſs, not only all the moſt remarkable Properties of the Conic-Sections, which are ſcattered up and down in moſt other Authors, but I have alſo inſerted ſeveral remarkable new ones, and have endeavoured to demonſtrate the whole with clearneſs and conciſeneſs.

In the Plates; I have, as much as it was poſſible, always marked the ſame lines with the ſame letters; and expreſſed them in the Book by the ſame *algebraical* Symbols, ſo that there are not above five or ſix letters to be remembered for the reading of the whole firſt Book. The ſame Definition ſerves for the three *Conic-Sections* (for the Circle and the Ellipſis are conſidered but as one), and the ſame Demonſtration ſerves likewiſe for each, except in a few caſes in the *Parabola*, wherein a particular Demonſtration is given, in order to render them the eaſier to Learners. I have taken particular care to demonſtrate ſuch Propoſitions, as relate to our Subject, which Sir *Iſaac Newton* (in his *Principia*) ſuppoſes the Reader to be acquainted with; and I have inſerted a Table of his Propoſitions, and the place in this Book where they are demonſtrated.

I ſuppoſe our Reader to have ſome knowledge both of the Elementary *Geometry* and common *Algebra*, and I adviſe Learners to read at firſt the Demonſtration for each *Conic-Section* ſeparately, and then conſider them all together. The *Ellipſis* and the *Hyperbola* have ſo great a connection, that their Equations (with reſpect to their Diameters) differ only in their Signs, for which reaſon the ſame Equations relating to both have a double Sign, the upper one belongs always to the *Ellipſis*.

As the intent of the Poſtulatum at the beginning, is only to ſhew the connection between the *Parabola* and the other Sections, it may be paſſed over, there being no Demonſtration depending thereon.

Tho' it may ſeem in a manner needleſs to treat of the Doctrine of *Fluxions*, ſince its Inventor, Sir *Iſaac Newton*, has already ſo fully delivered the Elements thereof, and ſo ſucceſsfully applied that admirable Diſcovery to the moſt ſublime parts of the Mathematics; yet becauſe moſt (if not all) Inventions of this Nature may at firſt appear ſomewhat abſtruſe, and perhaps ſeem to require ſome Illuſtration, I hope therefore that the Method I have taken in explaining this Doctrine, will not appear entirely uſeleſs. I make no uſe of infinitely ſmall quantities, nor of naſcent or evaneſcent velocities; and yet I think to have explained thoſe Principles, ſo that any Perſon of a moderate capacity, reading that part of this Book with due attention, may be fully convinced of the Truth thereof.

In order to this, I conſider a Curve-line to be generated by the motion of a point, urged by two powers acting in two different Directions, the one parallel to the Abſciſſes and the other parallel to the Ordinates. I prove from thence, that if this point (when arrived at a given place) did continue to move with the velocity it has there, it would proceed in a right-line touching the Curve in that place: From whence I conclude that the Direction of the point in the ſaid place, is in the Tangent: So that the three Directions being known in each place, the proportion between the velocities of the urging powers is likewiſe known.

Now the velocity in the Direction of the Abſciſſes being to the velocity in the Direction of the Ordinates, as the Subtangent is to the correſpondent Semi-ordinate; if the Law of theſe velocities is given, the Nature of the Curve may be found; and on the contrary, the Nature of the Curve being given, the Law obſerved by thoſe velocities may likewiſe be found.

In the uſe of *Fluxions* I conſider only finite quantities; and the Fluxion of a Rectangle, whoſe ſides are variable or flowing quantities, is found without rejecting any thing. Particular care has been taken in the ſecond Section of the ſecond Book to diſtinguiſh the *Maximums* from the *Minimums*, which has not been ſo much taken notice of as it requires: And for the better explanation of this uſeful Doctrine, I have illuſtrated it in the laſt Section of this Work, with ſeveral Problems, and every where clearly proved, that the *Maximum* or the *Minimum* required, has been really found. I muſt add in this place, that in *Article* 181, ſometimes the *Minimum* between two *Maximums* becomes nothing; eſpecially, when one is poſitive and the other negative; and ſometimes the *Maximum* between two *Minimums* becomes infinite.

In the third and laſt Book, general *Formula's* for the Fluxions of Areas or Surfaces, and Solids, are deduced from the ſame Principles as thoſe of Fluxions in general; and I hope in a clearer manner than is uſual; and likewiſe the general *Formula's* for finding the Centers of Gravity, Oſcillation and Percuſſion.

When the Reſolution of Problems depends on ſuch fluxionary Expreſſions, whoſe Fluents may be referred to the quadrature of the Conic-Sections, I have made uſe of the Tables of Sines and Logarithms, which Method is eſteemed to be ſhort and eaſy: And I have endeavoured to make the Principles thereof as clear as poſſible; and the more ſo, becauſe its uſe is almoſt entirely neglected, for want of having it ſufficiently exemplified; and beſides what we have here done, will fully explain and illuſtrate Mr. *Cotes's* TABLES of FLUENTS.

The laſt Section of this Treatiſe, relates to practical matters only: It contains a Select Collection of *Phyſico-Mathematical* Problems, the moſt part of which are very uſeful and wholly new. I have conſidered them in the moſt general manner that I could think of; ſo that ſeveral of them may ſerve as univerſal Canons for an infinite number of particular Examples: and as to thoſe which I have taken from other Authors, I have made their Conſtructions more ſimple and the Proceſs much ſhorter.

* * *

CON-

THE
CONTENTS of this WORK.

BOOK I.

page
SECT. I. *Of the* Conic-Sections *conſidered in Plano.*
Sect. II. *Of the* Conic-Sections *conſidered in the Solid*——64

BOOK II.

SECT. I. *Of the firſt Principles of* Fluxions *in general* 71
Sect. II. *Of the Method of* Maximums *and* Minimums 81
Sect. III. *Of the* Radii *of* Evolution——————85
Sect. IV. *Of the* Radii *of* Refraction *and* Reflection ———94

BOOK III.

SECT. I. *General Rules for finding* Fluents ———— 101
Sect. II. *General Rules for finding the Areas or Surfaces and Solids, together with the Rectification of* Curve-lines——111
Sect. III. *Of the* Centers *of* Gravity——————137
Sect. IV. *Of the* Centers *of* Oſcillation *and* Percuſſion ——148
Sect. V. *Contains Phyſico-Mathematical* Problems, *viz.*
Prob. I. *To find the Curve of a* Vault *or* Arch, *ſuch that all the Vouſſoirs remain in equilibrium with each other* ——— 160
Prob. II. *To find the beſt poſition of* Supporters *or* Buttreſſes 166
Prob. III. *To find the beſt poſition of a* Brace *or* Bracket *ſuſtaining another Timber which is fixed with one End to an horizontal line* ————————————168
Prob. IV. *To find the inclination of a Plane, ſuch that a Cubic* Box *open at top being drawn uniformly along that plane, ſhall raiſe out of a Reſervoir the moſt Water that is poſſible in a given Time*——————————169
Prob. V. *To find the angle which the* Gates *of a* Sluice *ought to make with each other, in order to reſiſt againſt the preſſure of the Water, with the greateſt Force poſſible*————170

Prob. VI. *To find the Charge of an* Engine *moved by a* Fluid, *ſo as to produce the greateſt Effect poſſible* — 172
Prob. VII. *To find on what Curve a cylindrical Weight muſt role, ſo as to keep another weight revolving about a Center, as a* Draw-bridge, *every where in equilibrium* — 174
Prob. VIII. *To find the angle which the Sails of a* Wind-Mill *ought to make in order to turn with the greateſt ſwiftneſs poſſible* — 176
Prob. IX. *To find the angle which the* Rudder *ought to make with the Keel of a* Ship *in order to bring her about in the leaſt time poſſible* — 177
Prob. X. *To find the Expreſſions of the* Reſiſtance *of* Figures *moving in a Fluid: Exemplified in the* Triangle, Cone, Sphere Spheroid *and* Paraboloid — 178
Prob. XI. *To find the* Fruſtum *of a* Cone *which ſhall meet with the leaſt Reſiſtance poſſible* — 182
Prob. XII. *To find the* Solid *of leaſt Reſiſtance* — 183
Prob. XIII. *To find the* Curve *of ſwifteſt deſcent according to any Law of the velocity: applied to the Circle and the Cycloid* 186
Prob. XIV. *To find the Center of* Oſcillation *of a compound* Pendulum, *ſuch that it ſhall perform its vibrations in the ſhorteſt time poſſible* — 188
Prob. XV. *The Center of* Oſcillation *of a Sphere being given, to find the* Diſtance *of the Sphere from the point of Suſpenſion* — 189
Prob. XVI. *The common center of* Oſcillation *of a compound Pendulum being given, to find the* Diſtance *of each Body from the point of Suſpenſion* — 189
Prob. XVII. *To find the* higheſt *and the* loweſt point *of the Helice of* Archimede's *Screw; as likewiſe the Proportion between the acting power and the weight, lift up, by means of that Screw* — 190
Prob. XVIII. *To find the* Expreſſions *of the* Times, Velocities, *and* Spaces *moved over, by a Body paſſing over a Curve: illuſtrated by two Examples* — 192
Prob. XIX. *To find the* Expreſſions *of the* Times, Velocities *and* Spaces *moved over, by a Body in a Curve, with any centripetal Force* — 194

Prob.

page

Prob. XX. *To find the* Curve *or Section of a* Fuſee *of a* Clock *or* Watch, *ſo that the Spring ſhall always draw equably or with an equal degree of ſtrength or force* — 195

Prob. XXI. *To find the nature of the* Curve *which a perfect flexible line will make, if preſſed or drawn by powers according to any* Law *and* Directions: *applied to the Curve which it makes if filled up with any fluid, to that if preſſed by the Atmoſphere of the Air; to the* Catenaria, Velaria *and to ſome other Examples* — 196

A Propoſition of the Cycloid *and* Epicycloid, *with all the moſt remarkable and uſeful Properties of thoſe* Curves — 202

Prob. XXII. *To find the* Time *in which a Body, only preſſed by its gravity, would move over a* Cycloid — 205

Prob. XXIII. *To find the* Time *in which a Body would move over any* Epicycloid, *with a centripetal Force every where proportional to the diſtance from the center* — 206

Prob. XIV. *To find the* Nature *of the Curve thro' which a Body paſſes in the ſhorteſt time poſſible, being preſſed or urged with any centripetal force: with a general conſtruction* — 207

Prob. XXV. *To find the* Nature *of the Curve which a* Ray *of light deſcribes in a Medium, whoſe denſity at equal diſtances from a given center, is equal, and increaſes by approaching that Center according to any Law whatever,* with a general Conſtruction — 211.

Theorem, *containing a* Method *for conſtructing Tables of Fluents* — 213

Prob. XXVI. *To reduce any Fraction, free from radical Signs, into as many ſimple ones as there are roots in the Denominator* 320

Prob. XXVII. *To find the Numerators of ſingle Fractions from the given Denominator of the compound Fraction. to which they are equal* — 220

To this are added,

A TABLE of Fluents of the ſeveral Caſes of the following Forms, *viz.* I. $\frac{d\dot{z}z^{rn-1}}{e+fz^n}$, II. $\frac{d\dot{z}z^{rn+\frac{1}{2}n-1}}{e+fz^n}$, III. $d\dot{z}z^{rn-1}\sqrt{e+fz^n}$, IV. $\frac{d\dot{z}z^{rn-1}}{\sqrt{e+fz^n}}$, V. $d\dot{z}z^{rn+\frac{1}{2}n-1}\sqrt{e+fz^n}$, VI. $\frac{d\dot{z}z^{rn+\frac{1}{2}n-1}}{\sqrt{e+fz^n}}$, VII. $d\dot{z}z^{rn-1}\sqrt{e+fz^n+gz^{2n}}$, VIII. $\frac{d\dot{z}z^{rn-1}}{\sqrt{e+fz^n+gz^{2n}}}$.

The Numbers of the following Rows preceded by *Prop.* *Lem.* and *Schol.* expreſs Sir ISAAC NEWTON's *Propoſitions*, and thoſe preceded by *Art.* expreſs reſpectively the Articles of this Book where theſe Propoſitions are demonſtrated.

Prop.	20,	21,	22,	23,	24,	25,	26,	27,	48,	49,	50,	51,
Art.	16,	141,	133,	134,	135,	136,	137,	138,	352,	353,	356,	360,

Lem.	12,	13,	15,	16,	17,	19,	20,	21,	23,
Art.	44,	41,	17,	140,	95,	96,	97,	125,	124.

Schol. after *Prop.* 27. *Book* I. *Art.* 129. *Schol.* after *Prop.* 34. *Book.* II. *Art.* 326, 327.

N. B. Page 106, line 9, if inſtead of reducing each ſide of the Equation $\frac{m}{1+z}=\frac{a+2bz+3czz+\ \&c.}{1+az+bzz+cz^3\ \&c.}$ into an infinite Series by a continual diviſion, you multiply croſs ways, in order to have $m+maz+mbzz+mcz^3+$ &c. $=a+az+2bz+2bz^2+3czz+3cz^3$ &c:, and then you ſuppoſe $m=a$, $ma=a+2b$, $mb=2b+3c$, &c. the Demonſtration will become ſomething ſhorter.

Mathe-

A
Mathematical Treatiſe:
Containing a SYSTEM of
CONIC-SECTIONS,
WITH THE
Doctrines of FLUXIONS and FLUENTS fully Explained.

SECTION I.

Of Conic-Sections *conſider'd in Plano.*

POSTULATUM.

GRANT that two infinite quantities, differing from each other by a finite quantity, may be eſteemed equal.

This *Poſtulatum* is here of uſe only to ſhew the connection of the Conic-Sections; but whenever we make uſe of it in the demonſtration of any Propoſition, we ſhall give always another Demonſtration independent on it.

DEFINITIONS.

1.

Fig. 3. 5. IF the Rectangle APa of the parts of a given right line Aa, terminated by the interſection P of the perpendicular Mm, be always to the Square $\overline{PM}^2$ or $\overline{Pm}^2$ of the parts of that line terminated by the line Aa and by a Curve-line MAm, in the given ratio of Aa to another line $2p$; that is, if $APa : \overline{PM}^2 :: Aa : 2p$; the Curvilinear Figures MAm are call'd by the general name of *Conic-Sections*; becauſe the interſections of a Cone and a Plane variouſly inclined, will repreſent ſuch figures, as will be ſhewn hereafter.

2.

If the line Mm always falls between the points A, a, and $Aa >$

* Fig. 3. or $< 2p$; the Figure * MAm is call'd an *Ellipſis*; and when $Aa =$

* Fig. 2. $2p$ it is call'd a * *Circle*.

3.

Fig. 5. If the line Mm always falls without the points A, a, and $Aa >$ or $< 2p$; the figures MAm, Nan are call'd *oppoſite Hyperbola's*; and when $Aa = 2p$, the oppoſite Hyperbola's are call'd *Equilateral Hyperbola's*.

But if the line Aa be biſected in C, and the Square $\overline{CP}^2$ *plus* the Square $\overline{CA}^2$ be always to the Square $\overline{PL}^2$ or $\overline{Pl}^2$ in the given ratio of Aa to $2p$, that is, if $\overline{CP}^2 + \overline{CA}^2 : \overline{PL}^2 :: Aa : 2p$, and the line LP interſects any where the line Aa; the figures LBS, lbs are call'd *Conjugate Hyperbola's* to the former, as on the contrary the former are ſaid to be *Conjugates* to the latter.

N. B. Hereafter we ſhall always conſider the oppoſite Hyperbola's as the ſame Section.

4.

* By the Poſtulat. If Pa becomes infinite, ſo that it may be taken for * Aa; the proportion $AP \times Pa : \overline{PM}^2 :: Aa : 2p$, will become $AP \times Aa : \overline{PM}^2 ::$

* Fig. 6. $Aa : 2p$ or $2p \times AP = \overline{PM}^2$, and the Figure * MAM is then called a *Parabola*.

5.

Any line as Aa, biſecting ſeveral parallel lines as MN, mn, FIG. 9, 10.
terminating in a Conic-Section, is call'd a *Diameter*; and when it cuts thoſe lines at right angles, it is alſo call'd an *Axis*.

6.

The lines MN, mn, are call'd *Ordinates* to the diameter Aa; and the ordinate Bb biſecting its diameter, is call'd *Conjugate Diameter* to the firſt, and on the contrary, the firſt is ſaid to be *Conjugate* to the ſecond.

7.

The Parts AP, Ap of a diameter terminated by its ordinates and the Curve, are call'd *Abſciſſes*.

8.

The Point A or a, where the diameter meets the Curve, is call'd the *Vertex*.

9.

The given line expreſſed by $2p$ is call'd the *Parameter* of the Diameter Aa.

10.

The Point F or f in the Axis, thro' which an ordinate to the FIG. 3. 5, 6.
Axis equal to the parameter $2p$ paſſes, is call'd the *Focus*.

11.

And the point C which biſects the Axis Aa, is call'd the *Center*. FIG. 3. 5.

12.

A right line meeting a Conic-Section but in one point, and which continued both ways, does not fall within the Section, is call'd a *Tangent*.

N. B. Henceforward we ſhall always call the Abſciſs $AP = x$, $CP = u$, the Semi-ordinate PM or $PL = y$, the firſt Axis $Aa = 2a$, the ſecond or Conjugate $Bb = 2b$, and the diſtance between the foci $fF = 2d$.

COROL-

COROLLARY I.

FIG. 3. 5. 1. HEnce $AP \times PA = (xx\,\overline{2a \mp x})\; 2ax \mp xx$, or $AP \times Pa =$ $(\mp\overline{a \mp u} \times \overline{a + u})\; \pm aa \mp uu$. Therefore ſubſtituting the a-
* *Def.* 1. nalytical values into the proportion $APa : {}^{*}\overline{PM}^{2} :: Aa : 2p$; we ſhall
have $2ax \mp xx : yy :: 2a : 2p$, or $\frac{a}{p}yy = 2ax \mp xx$ in the firſt caſe;
and $\pm aa \mp uu : yy :: 2a : 2p$, or $\frac{a}{p}yy = \pm aa \mp uu$ or $\pm\frac{a}{p}yy = aa - uu$
* *Def.* 2. in the ſecond, *viz.* $\frac{a}{p}yy = 2ax - xx$, $\frac{a}{p}yy = aa - uu$, in the *
* *Def.* 3. Ellipſis; $\frac{a}{p}yy = 2ax + xx$, $\frac{a}{p}yy = uu - aa$ in the * Hyperbola
MAm, $\frac{a}{p}yy = uu + aa$ in the Conjugate Hyperbola's LBS; and
* *Def.* 4. $2px = yy$ in the * Parabola.

COROLLARY II.

FIG. 3. 5. 2. WHen $CP = u = o$, $PM = y$ will become $= CB = b$, in the Ellipſis, and $PL = y = CB = b$ in the Hyperbola: therefore the Equation $\frac{a}{p}yy = aa \mp uu$, becomes $bb = ap$: that is, *the parameter* $2p$ *of the firſt Axis* Aa *is a third proportional to the firſt and ſecond Axis in the Ellipſis and Hyperbola.*

COROLLARY III.

3. BEcauſe $ap = bb$, or $p = \frac{bb}{a}$: by ſubſtituting for p its value into the Equations $\frac{a}{p}yy = 2ax \mp xx$, $\pm\frac{a}{p}yy = aa - uu$, we ſhall have $\frac{aa}{bb}yy = 2ax \mp xx$ and $\pm\frac{aa}{bb}yy = aa - uu$.

COROLLARY IV.

FIG. 3. 5, 6. 4. WHEN $p = y$; the Abſciſs $CP = u$, will become equal to
* *Def.* 10. the diſtance $CF = d$ of the * Focus F or f, from the Center

Center C, in the Ellipſis and Hyperbola; therefore $\pm\frac{a}{p}yy = aa - uu$ will become $\pm ap = aa - dd$ or $dd = aa \mp bb$, becauſe $ap = {}^{*}bb$. * Art. 2. And the Abſciſs AP will become equal to the diſtance of the focus F from the vertex A, in the Parabola; which therefore is equal to the fourth part of the parameter $2p$, ſince $2px = yy$ becomes $x = \frac{1}{2}p$ in this Caſe.

Hence it follows, 1°. That if the two Axes of an Ellipſis or Hyperbola are given, the foci may be found, by making $bF = bf = AC$ in the Ellipſis, and $CF = Cf = AB$ in the Hyperbola; ſince the right angled triangle bCF or ACB, gives $(\overline{bF}^2)$ $\overline{AC}^2 - \overline{bC}^2 = \overline{CF}^2$ $(aa - bb = dd)$ in the Ellipſis, and $(\overline{AB}^2)$ $\overline{CF}^2 = \overline{CA}^2 + \overline{CB}^2$ $(dd = aa + bb)$ in the Hyperbola.

2°. That if the firſt axis Aa, and the diſtance *f*F between the foci are given, the vertex's of the ſecond or conjugate axis Bb may be found, by making $FB = Fb = AC$ in the Ellipſis, and $AB = Ab = CF$ in the Hyperbola.

3°. In the Ellipſis, the ſecond axis is always leſs than the firſt; ſince the Hypothenuſe bF (AC) of the right angled triangle bCF, is always greater than any one of its ſides: And when $bC = AC$, the Ellipſis will become a Circle.

4°. In the Hyperbola, the ſecond axis Bb may be either leſs, equal to, or greater than the firſt: When they are equal, the Hyperbola will be equilateral.

COROLLARY V.

5. IN the Ellipſis and Hyperbola, if the line MQ be drawn perpendicular to the ſecond axis Bb, then will $QM = CP = u$, $CQ = PM = y$; and becauſe $\pm\frac{a}{p}yy^{*} = aa - uu$, or * Art. 1. $p\frac{uu}{a} = ap \mp yy$, we ſhall have $ap \mp yy : uu :: p : a$, that is, $\overline{BC}^2 \mp \overline{CQ}^2 : \overline{QM}^2 :: p : AC$, becauſe $ap = {}^{*}bb$. * Art. 2.

COROLLARY VI.

6. IF from any point as N of the Curve, the lines Np, Nq, are drawn perpendicular to the axes; we ſhall have APa:

$\overline{\text{PM}}^2 :: \text{Apa} : \overline{\text{pN}}^2 :: \text{AC} : p$, and $\overline{\text{BC}}^2 \mp \overline{\text{CQ}}^2 : \overline{\text{QM}}^2 :: \overline{\text{BC}}^2 \mp \overline{\text{Cq}}^2 : \overline{\text{qN}}^2$
FIG. 6. $:: p : \text{AC}$. And in the Parabola $2p \times \text{AP} : 2p \times \text{Ap}$ or $\text{AP} : \text{Ap} :: \overline{\text{PM}}^2 : \overline{\text{pm}}^2$, that is, *the Squares of the Ordinates are as their correſpondent Abſciſſes.*

COROLLARY VII.

FIG. 5. 7. IF from the point L, the line LR be drawn perpendicular to the axis B b; then will $\text{PL} = \text{CR} = y$, $\text{CP} = \text{RL} = u$:
* *Art.* 1. and becauſe * $\frac{a}{p}yy = uu + aa$, or $yy - ap = p\frac{uu}{a}$, we ſhall have $\overline{\text{CR}}^2 - \overline{\text{CB}}^2 : \overline{\text{RL}}^2 :: p : a$, becauſe $ap = bb$. Which ſhews that the Hyperbolas LBS, lbs, are the ſame as their Conjugates MAm, Nan, except Bb is their firſt axis, and Aa their ſecond.

COROLLARY VIII.

FIG. 9, 10. 8. IT is evident, that in the Ellipſis and oppoſite Hyperbola's, the Ordinates to either of the axes, equally diſtant from the Center C, are equal.

For if $\text{CP} = \text{Cp} = u$; PM or $\text{PN} = y$ will be equal $=$ pm or pn, ſince $\pm \frac{a}{p}yy = aa - uu$.

COROLLARY IX.

9. BEcauſe $\frac{a}{p}yy = aa - uu$ in the Ellipſis; it is evident, that when CP (u) increaſes, PM (y) decreaſes, till CP becomes $=$ CA or Ca, where $\text{PM} = 0$. Which ſhews, that the Elliptical-Curve recedes more and more from the vertex A of the firſt axis, thro' which it paſſes, till it meets the ſecond axis Bb, afterwards it approaches continually, till it meets the firſt axis again in the vertex a.

And becauſe $\frac{a}{p}yy = uu - aa$ in the Hyperbola, when CP(u) increaſes, PM (y) increaſes alſo, and when $\text{CP} = \text{CA}$, then $\text{PM} = 0$: which ſhews, that the Hyperbolical-Curves paſs thro' the vertex's A, a, and afterwards recede from their firſt axis *ad infinitum.*

Becauſe $2px=yy$ in the Parabola; when AP (x) increaſes, PM (y) increaſes likewiſe, and when AP $=0$, then PM is alſo $=0$: therefore the Parabolical-Curve paſſes thro' the vertex A, and recedes from its axis *ad infinitum.* FIG. 6.

COROLLARY X.

10. A Line ef, drawn thro' the vertex A of any diameter Aa of a Conic-Section, parallel to its ordinates MN, touches the Curve in that point. For PM being always equal to PN, it is plain, that when MN falls in A, both the ſemi-ordinates PM, PN, become equal to nothing. FIG. 9, 10.

THEOREM I.

11. THE *Sum in the Ellipſis, or the difference in the Hyperbola, of the lines* FM, fM, *drawn from the foci to any point of the Curve, is always equal to the firſt axis, that is,* $f\text{M} \pm \text{FM} = \text{A a}$. FIG. 3, 5.

The right angled triangle fMP gives, $\overline{f\text{M}}^2 = dd + 2du + uu + (yy) {}^* \pm bb \mp \frac{bb}{aa}uu = dd \pm bb + 2du + \frac{aa \mp bb}{aa} \times uu$, or becauſe $dd = {}^* aa \mp bb$, $\overline{f\text{M}}^2 = aa + 2du + \frac{dd}{aa}uu$, and $f\text{M} = {}^* a + \frac{du}{a}$. And the right angled triangle FPM gives, $\overline{\text{FM}}^2 = dd - 2du + uu + (yy) \pm bb \mp \frac{bb}{aa}uu$ or $\text{FM} = aa - 2du + \frac{dd}{aa}uu$, and $\text{FM} = \pm a \mp \frac{du}{a}$. Therefore $f\text{M} \pm \text{FM} = 2a$. W. W. D.

* *Art.* 3.
* *Art.* 4.

In the Parabola, $\text{FM} = \text{AP} + \text{AF}$. For $\text{FP} = (\pm \text{AP} \mp {}^* \text{AF}) \pm x \mp \frac{1}{2}p$, and $\overline{\text{FM}}^2 = xx - px + \frac{1}{4}pp + (yy) 2px$ or $\overline{\text{FM}}^2 = xx + px + \frac{1}{4}pp$. Therefore $\text{FM} = x + \frac{1}{2}p$. FIG. 6.

* *Art.* 4.

COROLLARY I.

2. BEcauſe $\overline{f\text{M}}^2 + \overline{\text{FM}}^2$ $(2dd + 2uu + 2yy) = 2\overline{\text{CF}}^2 + 2\overline{\text{CM}}^2$, the Sum of the ſquares of the two ſides fM, FM, of any triangle fMF, is always equal to double the Sum of the FIG. 3. 5.

Squares of half its baſe $\overline{CF}^2$ or $\overline{Cf}^2$ (dd), *plus* the Square $\overline{CM}^2$ $(uu+yy)$ of the line drawn from its vertex M to the middle of the baſe ; and the difference of thoſe Squares $\overline{fM}^2 - \overline{FM}^2$, will be $= (4du)\ 4CP \times CF$.

SCHOLIUM.

Hence the Sum *of the Squares of two lines drawn from any point, to the extremities of a diameter of a Circle or Sphere, or to two oppoſite angles of a Square, Rectangle, Cube or Parallelepipedon, will be always equal to the* Sum *of the Squares of two other lines, drawn from that point to the extremities of any other diameter, or to any other two oppoſite angles.*

COROLLARY II.

Fig. 11, 12. 13. IN the Ellipſis or Hyperbola, if the point B be taken in the firſt axis aA, ſo that $\div CF : CA : CB$, and from any point of the Curve as M and the point B, be drawn MP, BL perpendicular, and ML parallel to AC; then will, MF : ML :: CF : CA. For $MF = {}^*\pm a \mp \frac{du}{a}$, and $ML = PB = \pm \frac{aa}{d} \mp u$ by Conſtr.; therefore $\pm a \mp \frac{du}{a} : \pm \frac{aa}{d} \mp u :: d : a$. Since the product of the means, and that of the extreams gives, $\pm aa \mp du = \pm aa \mp du$. * Art. 11.

When the point M coincide with the vertex A, then will AF : AB :: CF : CA :

Fig. 13. In the Parabola, if AB = AF ; then will ML = MF, ſince $ML = PB = x^* + \frac{1}{2}p$. * Art. 11.

DEFINITION 13.

F. 11,12,13. THE Line LB is call'd the *Directrix* of thoſe Curves.

COROLLARY III.

14. HEnce if the line Ae, be perpendicular to the axis Aa and equal to AF, and the Semi-ordinate PM be produced ſo as to meet in G, the line drawn thro' the points B, e ;

then will FM be always = PG. For becauſe of the Similar triangles BAe, BPG, we ſhall have, AB:Ae or AF::PB or ML :PG. But AB:AF::*ML:MF. Therefore PG = MF. * *Art.* 13.

COROLLARY IV.

15. FRom whence may be reduced a very eaſy *Method* for deſcribing a Conic-Section, when the firſt axis and the foci are given.

For if you draw a great number of lines, interſecting the axis at right angles, and you make Ae = AF; then if the line Be be drawn, and FM be made always equal to its correſpondent PG; the Curve-line, paſſing thro' all the points M, will determine the Section required.

N. B. If AB be > AF, the Section will be an Ellipſis; if AB < AF, an Hyperbola; and if AB = AF, a Parabola.

COROLLARY V.

16. IF the focus F, the ratio $\frac{n}{m}$ of the firſt axis Aa to the diſtance between the foci F, f, and two points M, N, of the Curve are given, the axis and the other focus f may be found in the following manner.

Draw FM, FN, take ML to MF and NE to NF, in the given ratio of n to m, and from the Centers M, N, with the radii ML, NE, deſcribe two circular arches ur, ns: then if thro' the focus F, you draw aB at right angles to LE (a tangent to thoſe arches), and make AF: AB::MF: ML; the point A will be * the vertex. If therefore CF = v, AF = c, and AC = $\frac{n}{m}v$; * *Art* 13. then will (CF ± AF = AC) $v \pm c = \frac{n}{m}v$, or $\frac{\pm mc}{n-m} = v$.

N. B. When $n > m$, the Section will be an Ellipſis; when $n < m$, an Hyperbola; and when $n = m$, a Parabola.

Several Conſtructions of the *Conic-Sections*, when the firſt Axis and the Foci are given.

For the Ellipſis.

FIG. 3. HAving faſten'd the ends of a thread *f*MF equal to the firſt axis, in the foci *f*, F; the Style or Pin M, which ſerves to keep the thread always tight, will deſcribe * in its motion about the foci, the Curve of the Ellipſis required. * *Art.* 11.

Otherwiſe.

FIG. 4. Let two Rulers AF, B*f*, each of them equal in length to the firſt axis, have one of their ends faſten'd in the foci F, *f*, and the others A, B, to a third Ruler AB = to F*f*, ſo that they may freely turn about the four points A, B, F, *f*; then the interſection M will be always in the Curve.

For becauſe AB = F*f*, AF = B*f*, the triangles A*f*F, *f*AB will be ſimilar and equal, alſo the angles A*f*B, FA*f* are equal; therefore *f*M = MA, and FM + *f*M will always be = FA.

For the Hyperbola.

FIG. 5. Let one of the ends of a Ruler D*f*, and of a thread FMD, (whoſe difference in length is equal to the firſt axis Aa) be faſten'd in the foci *f*, F, and the other end of the thread to that of the Ruler D: the Style or Pin M, which keeps the thread always tight, and its part DM always cloſe to the edge of the Ruler, will deſcribe by its motion * the Curve MAm of the Hyperbola required. * *Art.* 11.

In the ſame manner, if the end of the Ruler be faſten'd in the focus F, and the end of the thread in the focus *f*, (retaining the ſame length) you may deſcribe, the Curve Nan of the oppoſite Hyperbola to the firſt.

FIG. 1. *N. B.* When the length of the Ruler is equal to the length of the thread, the Style or Pin M, will deſcribe a right line CM.

For

For the Parabola.

Fig. 6. Faſten one end of a thread FMf in the focus F, and the other to the end f of a Square fDE, whoſe ſide Df is parallel to the axis, and equal in length to the thread: The Style or Pin M, which keeps the thread always tight, and its part fM cloſe to the edge of the Square, will deſcribe (by ſliding the Square along a Ruler EE) the Portion MAM of the Parabolical Curve required. For MF will be always $= MD = PB =$ * $AP + AF$. * *Art.* 11.

Otherwiſe.

Draw a great number of lines MM, mm, interſecting the axis AP at right angles in P, p, take $AB = AF$, and make always MF equal to its correſpondent PB; then the Curve-line drawn thro' all the points M, m, will be a * Portion of the Curve of the Parabola required. * *Art.* 13.

Otherwiſe for the Ellipſis or Hyperbola.

Fig. 7, 8. Take the line fD equal to the firſt axis, and from the focus f as a Center with the radii fN, fN, deſcribe a great number of circular arches MNM interſecting the line fD in N, N,: then if the lines FM, FM are made always equal to their correſpondent DN's; the Curve-line, paſſing thro' all the points M, M, will determine the Section required. For $fM = fD \mp DN$, or $(fM \pm DN)$ $fM \pm FM = fD =$ to the firſt axis.

PROBLEM I.

17. *To draw a Line, which ſhall touch a Conic-Section in a given point* M.

F. 14,15,16. From the foci F, f † draw the lines FM, fM, and take $ML = MF$; then the line MD perpendicular to the line FL joining the points F, L, will be the Tangent required. For if lines are drawn, from the foci and from the point L to any other point m of the

---

† In the Parabola, the point or focus is taken any where in the line LM, which paſſes thro' the point of contact M parallel to the axis.

line MD; then will Fm = Lm, becauſe DL = DF; therefore fm ± FM or fm ± Lm will be always greater than fL = fM
* *Art.* 11. ± FM = * 2a; and ſince there is but one point M in the line MD, from which two lines may be drawn to the foci, ſuch that their Sum in the Ellipſis and Parabola, or their difference in the Hyperbola, be equal to the given line 2a; it follows that the line MD only touches the Curve in M. *W. W. D.*

COROLLARY I.

18. IT is manifeſt, that the angles fMt, FMT, made by any tangent, and by the lines drawn from the foci to the point of contact, are equal: Since FM = ML, and FD = DL, the angle FMD will be equal to the angle DML = fMt.

COROLLARY II.

* *Art.* 11. 19. BEcauſe fM = * $a+\frac{du}{a}$, in the Ellipſis and Hyperbola, and fL = 2a, fF = 2d. If from the point of Contact M, the line MK be drawn at right angles to the tangent, interſecting
* F. 14, 16. the Axes in K, k; then * will fL : fF :: fM : fK = $\frac{aad+ddu}{aa}$.
* *Art.* 4. Therefore ± fP ∓ fK = KP = $\frac{\pm aa\mp dd}{aa}\times u$ = * $\frac{bb}{aa}u=\frac{pu}{a}$, becauſe $\frac{bb}{a}=p$.

COROLLARY III.

20. HEnce, if the tangent MD be produced ſo as to meet the axes in T, t; we ſhall have, 1.° KP $\left(\frac{pu}{a}\right)$: PM (y) :: PM
* *Art.* 1. (y) : PT = $\frac{ayy}{pu}=\frac{\pm aa\mp uu}{u}$; becauſe * $\frac{a}{p}yy=\pm aa\mp uu$. 2.° CP : aP :: AP : PT. $\left(u:a+u::\pm a\mp u:\frac{\pm aa\mp uu}{u}\right)$; becauſe CP × PT = aP × AP. 3.° ÷ CP : CA : CT; becauſe CP ± PT = $\frac{aa}{u}$ = CT. 4.° CP : CA :: AP : AT; ſince ± CT ∓ CA = AT = $\frac{\pm aa\mp au}{u}$. 5.° TA : Ta :: PA : Pa : $\left(\pm\frac{aa}{u}\mp a:\frac{aa}{u}+a::\pm a\mp u\right.$ $\left.:a+u\right)$,

$:a+u)$, becauſe $TA \times Pa = Ta \times PA$. 6.° And $TP \times PC = APa$, ſince $PT = \frac{\pm aa \mp uu.}{u}$

COROLLARY IV.

21. IF from the point of contact M, be drawn the Semi-ordinate Mp to the ſecond axis; we ſhall have $PK : PM :: pM : pk = \frac{a}{p}y$. And $pk : pM :: pM : pt = \frac{puu}{ay} = \frac{ap \mp yy}{y}$, becauſe $\pm\frac{a}{p}yy = aa - uu$. Therefore $pt \pm Cp = Ct = \frac{ap}{y} = \frac{bb}{y}$, becauſe * $p = \frac{bb}{a}$. * Art. 2.

Whence it appears, that the properties of the Tangents are the ſame with regard to both the Axes.

COROLLARY V.

22. BEcauſe $TA : Ta :: PA : Pa$. When Pa becomes infinite and alſo = * Ta, it is evident, that TA becomes = PA; * By the Poſtulat. and the Ellipſis then becomes a Parabola. FIG. 14, 15.

Otherwiſe.

Since the angles FTM, FMT (= fMt) are equal; FM = * x * Art. 11. $+ \frac{1}{2}p$ will be = FT. But $AF = \frac{1}{2}p$; therefore $FT - AF = x =$ FIG. 15. AT. From whence it follows:

1.°

23. Becauſe of the right angled ſimilar Triangles KPM, MPT, we ſhall have $TP : PM :: PM : PK = \frac{yy}{2x} = p$, ſince $2px = yy$.

2.°

24. The Tangent AD drawn thro' the vertex A, will meet the tangent MT in the ſame point, as the line FD drawn from the focus perpendicular to MT. For becauſe FM = FT; FD

biſects MT. And ſince PM, AD are parallel, and AT = AP; AD does likewiſe biſect MT.

3.°

25. The perpendicular FD is a mean proportional between FA and FM; ſince FA:FD::FD:FT = FM.

COROLLARY VI.

Fig. 17, 18. 26. IN the Ellipſis and Hyperbola, if the line CN be drawn parallel to the Tangent MT, and from its interſection N with the Curve, and from the point of contact M, be drawn the Semi-ordinates NQ, MP to the axis Aa; then will $\overline{CP}^2 = \overline{AC}^2 \mp \overline{CQ}^2$. For if $CQ = v$, $QN = z$, the ſimilar triangles TPM, CQN will give, $(\overline{TP}^2 : \overline{PM}^2 :: \overline{CQ}^2 : \overline{QN}^2)$ * $\frac{\overline{\pm aa \mp uu}^2}{uu} : (yy) \frac{p}{a} \times \overline{\pm aa \mp uu} :: vv : (zz)$ * $\frac{p}{a} \times \overline{aa \mp vv}$, or $\pm aa \mp uu : uu :: vv : aa \mp vv$; and by compoſition in the Ellipſis, or by diviſion in the Hyperbola, $aa : uu :: aa : aa \mp vv$. Therefore $uu = aa \mp vv$.

* Art. 20.

* Art. 1.

COROLLARY VII.

27. HEnce $\pm \frac{a}{p} yy = (aa - uu) \pm vv$, or $\frac{a}{p} yy = vv$; and $\frac{a}{p} zz = (aa \mp vv) uu$, or $\frac{a}{p} zz = uu$. And becauſe $p = \frac{bb}{a}$; $\frac{a}{p} yy = vv$ will become $\frac{aa}{bb} yy = vv$, or $\frac{a}{b} y = v$, and $\frac{az}{b} = u$. Likewiſe $\frac{ay}{b} \times u = \frac{az}{b} \times v$, or $uy = vz$: and $\frac{aa}{bb} \times \overline{zz \pm yy} = uu \pm vv =$ * aa, or $zz \pm yy = bb$.

* Art. 26.

COROLLARY VIII.

28. IF the lines ſd, FD, are drawn from the foci perpendicular to the Tangent MT; the Rectangle ſd × FD will be equal to the Square $\overline{BC}^2$ of the Semi-ſecond axis. For let the line DC meet dſ, produced if neceſſary, in E; then becauſe Cſ = CF; CD

will be $= CE$, and $fE = FD = {}^{*}DL$: alſo DE will be parallel * *Art.* 17.
and equal to $fL = {}^{*}Aa$. Therefore the circular circumference * *Art.* 17.
deſcribed from the center C with the radius CA, will paſs thro'
the points E, a, d, D, A; conſequently $Ef \times fd$, or becauſe $Ef =$
FD, $fd \times FD = afA = {}^{*}\overline{BC}^2$. * *Art.* 4.

COROLLARY IX.

29. IF the lines fM, FM, are drawn from the foci to any point FIG. 19, 20.
of the Curve, and the lines CN, FL, parallel to the tangent
at M, interſecting fM in E, L; then will $EM = AC$. For fC being $= CF$; fE will be $= EL = \frac{fM \mp ML}{2}$. But becauſe the
angles made by the lines fM, FM and the tangent, are * equal; * *Art.* 18.
the triangle MLF will be iſoſceles: alſo $ML = MF$. Therefore
$EL \pm ML = {}^{*}AC = EM$. * *Art.* 11.

COROLLARY X.

30. IF Cr be drawn perpendicular to the tangent TM; then will FIG. 17, 18.
$BC \times AC = CN \times Cr$. For $NC : NQ\,(z) :: CT\,{}^{*}\left(\frac{aa}{u}\right) : Cr$, * *Art.* 20.
or $CN \times Cr = \frac{z}{u}aa = ab$, becauſe $z = {}^{*}\frac{bu}{a}$. * *Art.* 27.

COROLLARY XI.

31. IF $fM = r$, $FM = s$; then will $r \pm s = {}^{*}2a$; and $fd =$ * *Art.* 11.
$\frac{br}{\sqrt{rs}}$, $FD = \frac{bs}{\sqrt{rs}}$, $Cr = \frac{ab}{\sqrt{rs}}$. For $fM : fd :: FM : FD =$
$\frac{fd \times FM}{fM}$, and $\overline{BC}^2 = {}^{*}fd \times FD$; therefore $\overline{FD}^2 = \frac{\overline{BC}^2 \times FM}{fM} = \frac{bbs}{r}$ * *Art.* 28.
$= \frac{bbss}{rs}$, or $FD = \frac{bs}{\sqrt{rs}}$, and $fd = \frac{FD \times fM}{FM} = \frac{br}{\sqrt{rs}}$. But becauſe Cr,
Ed are parallel and $DC = \frac{1}{2}DE$; Cr will be $= \frac{1}{2}Ed = {}^{*}$ * *Art.* 28.
$\frac{fd \pm FD}{2} = \frac{br \pm bs}{\sqrt{rs}} = \frac{ab}{\sqrt{rs}}$. Alſo $CN = \sqrt{rs}$ or $\overline{CN}^2 = fM \times FM$,
becauſe $CN \times Cr = {}^{*}ab$. It follows likewiſe, that $FM \times BC =$ * *Art.* 30.
$FD \times CN$, and $fM \times BC = fd \times CN$.

COROLLARY XII.

32. IF thro' the point of Contact M (the reſt as before) the
Fig. 21, 22. line MK be drawn at right angles to the tangent MT,
and meeting the axes in K, k; then will FK $= \frac{ds}{a}$, and fK $= \frac{dr}{a}$.
For fL : fF :: (ML) MF : FK $= \frac{ds}{a}$ :: fM : fK $= \frac{dr}{a}$. And
MK $= \frac{b}{a}\sqrt{rs}$, Mk $= \frac{b}{p}\sqrt{rs}$: becauſe fL : LF ($=$ 2FD) :: fM :
* Art. 19. MK $= \frac{b}{p}\sqrt{rs}$, and KP* $\left(\frac{pu}{a}\right)$: PC (u) :: MK : Mk $= \frac{b}{p}\sqrt{rs}$.

But if from the points K, k, the lines kg, Ka, are drawn at right angles to either of the lines fM, or FM; we ſhall have M$a = p$, in the three Conic-Sections, and Mg $= a$, in the Ellipſis and Hyperbola. For fM : fd :: MK : Ma :: Mk : Mg.

COROLLARY XIII.

Fig. 23, 24. 33. IF the tangents (the reſt as before) drawn thro' the vertex's A, a, of the firſt axis, meet the tangent TM, in
* Art. 20. E, e; then will AE × ae $= \overline{BC}^2$. For TP : PM :: *$y : \frac{pu}{a}$ :: Ta
$\left(\frac{aa + au}{u}\right)$: ae $= \frac{a + u}{y} \times p$ :: TA $\left(\frac{\pm aa \mp au}{u}\right)$: AE $= \frac{\pm a \mp u}{y} \times p$.
Therefore AE × ae $= pp \times \frac{\pm aa \mp uu}{yy} = ap = bb$, becauſe $p = \frac{bb}{a}$,
* Art. 1. and $\frac{a}{p}yy =$ * $\pm aa \mp uu$.

COROLLARY XIV.

* Art. 4. 34. BEcauſe AE × ae $= \overline{BC}^2$, and ($\pm aa \mp dd =$ *bb) AF × Fa
* Art. 28. $= \overline{BC}^2 =$ *FD × fd; it follows that AE × ae $=$ AFa $=$
* Art. 20. FD × fd. Likewiſe TE × Me $=$ Te × ME, ſince * TA : Ta ::
PA : Pa.

THEOREM

THEOREM II.

35. *ANY line as* Mn *paſſing thro' the center of an Ellipſis or Hyperbola and terminated by the Curve, is biſected by the ſaid Center.* FIG. 9, 10.

For if $CP = Cp$, the ſemi-ordinates PM, pn, to the axis Aa, and alſo the ſimilar triangles CPM, Cpn, will be equal. Therefore $CM = Cn$: and becauſe CP, Cp, are in a right line; CM, Cn, will be alſo in a right line. *W. W. D.*

COROLLARY I.

36. HEnce, any line as Mm terminated by the Curve of an Ellipſis or Hyperbola, and parallel to the line Aa paſſing thro' the center, is biſected by the line Bb paſſing likewiſe thro' the center, parallel to the tangent *fe* in A. FIG. 25, 26. For if the line mN drawn thro' the center meets the Curve in N; the line joining the points M, N, will be biſected by the line Aa, and alſo parallel to the * tangent *ef*, or parallel to the line Bb by Hyp. * *Art.* 10. Therefore, becauſe $mC = CN$; mQ will be always $= QM$.

When the center C of the Ellipſis removes at an infinite diſtance, it is evident that its diameters become then parallel between themſelves; but the Section will then be a Parabola, and the diameters will ſtill biſect their Ordinates, as it eaſily may be prov'd from the 3d Prop. From whence it follows: FIG. 26.

That all the Diameters of an Ellipſis or Hyperbola, paſs thro' and are biſected by the Center; and that in the Parabola, they are all parallel to the axis.

COROLLARY II.

37. BEcauſe each diameter biſects its ordinates; it follows, that a line biſecting two parallel lines terminating in a Conic-Section, will be a diameter.

COROLLARY III.

38. HEnce, in a given Conic-Section, may be drawn a Diameter, which ſhall make a given angle with its ordi- FIG. 27, 28.

nates. For if in the Ellipſis or Hyperbola, you deſcribe upon a given diameter m M as a chord, a circular arch M L m, ſo as to contain the given angle: the line A a or B b biſecting the chord M L or L m, drawn thro' the interſection of that arch and the Curve, and paſſing thro' the center C of the Section, will be the diameter required: Since M L is an ordinate to the diameter A a, and the angle M P C is equal to the given angle M L m.

Fig. 29. But in the Parabola, draw the line M L ſo as to make with the given diameter M m, the given angle; then the line A C parallel to M m, will be the diameter required.

THEOREM III.

Fig. 31, 32. 39. *If the line* P M *be a ſemi-ordinate to the diameter* A a *conjugate to* B b, *that is, if* P M *be parallel to the tangent* A T: *I ſay, the Proportion* $\overline{PM}^2 : APa :: \overline{BC}^2 : \overline{AC}^2 a : p : AC$, *expreſſes likewiſe the properties of any two conjugate Diameters with regard to their ordinates.*

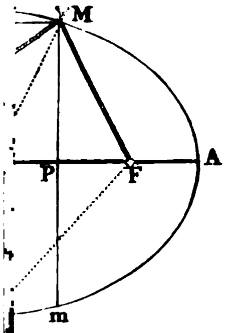
From the points A, P, M, B, draw the lines A S, P D, M Q, B R, perpendicular, and P E parallel to the axis N n, and make $CD = r$, $PE = s$, $CS = u$, $SA = y$. Then becauſe of the ſimilar triangles T S A, P E M, and C D P, C S A, we ſhall have

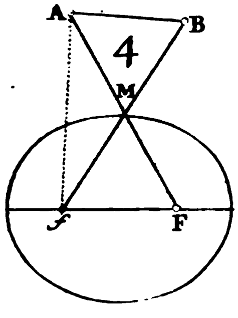
* Art. 20. $TS : SA :: y : p\frac{u}{a} :: PE : EM = p\frac{us}{ay}$, $DP = QE = \frac{r}{u}y$: alſo $QM = QE + EM = p\frac{us}{ay} + \frac{r}{u}y$, $CQ = PE \mp CD = s \mp r$. Now ſubſtituting theſe analytical values into the Equation of the Curve

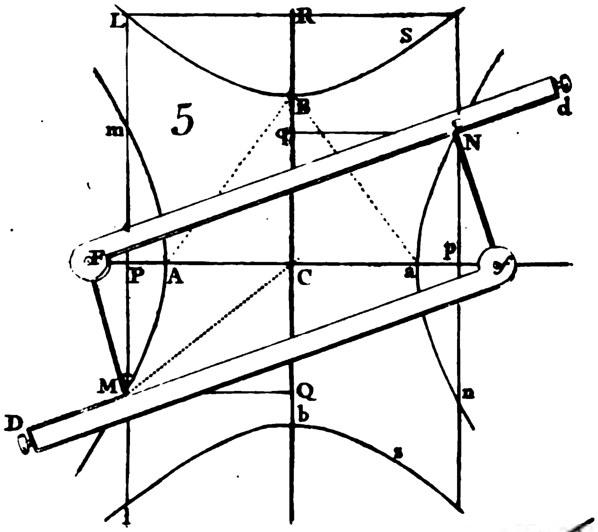
* Art 1. $* \pm \frac{a}{p}\overline{QM}^2 = \overline{CN}^2 - \overline{CQ}^2$ it will become $\pm p\frac{uuss}{ayy} \pm \frac{arr}{puu}yy = aa - ss - rr$, and ſubſtituting for uu in the firſt term, and for yy in the ſecond, their values taken in the Equation $\pm \frac{a}{p}yy = aa - uu$, we ſhall have $\pm p\frac{ss}{ayy} + \frac{rr}{uu} = 1$, after the reduction done.

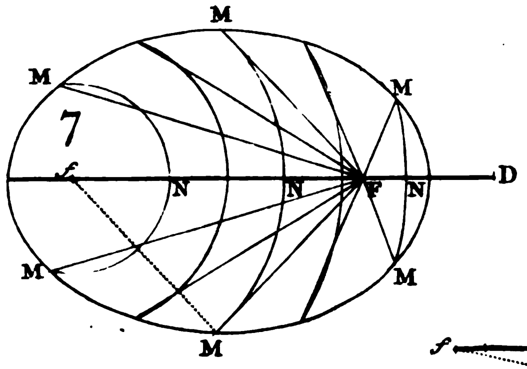
* Art. 27. Therefore $\overline{PE}^2 : \pm \overline{CS}^2 \mp \overline{CD}^2 :: \overline{CR}^2 \left(* \frac{a}{p}yy\right) : \overline{CS}^2$; or becauſe of

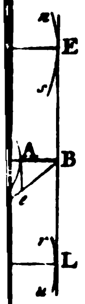
* Art. 2. the ſimilar triangles, $\overline{PM}^2 : \pm \overline{CA}^2 \mp \overline{CP}^2$ or $APa :: \overline{CB}^2 : \overline{CA}^2 :: * p : AC$.

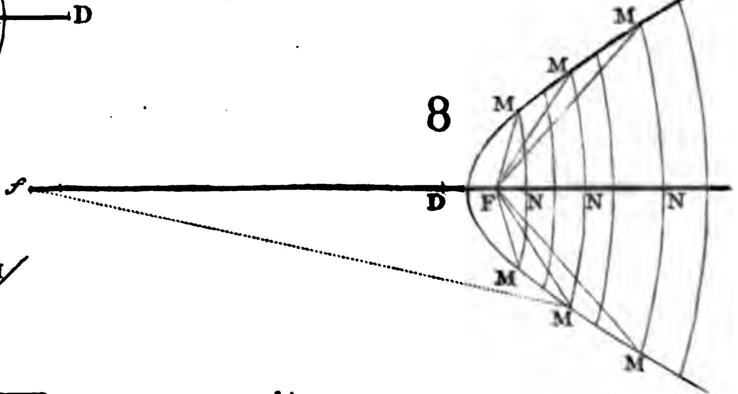

M
P
F
A
m

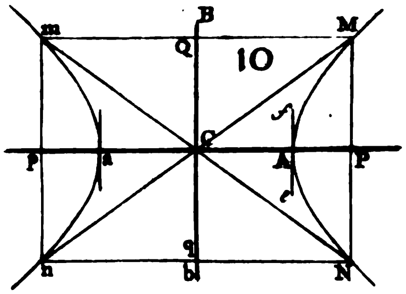

A
B
4
M
f
F

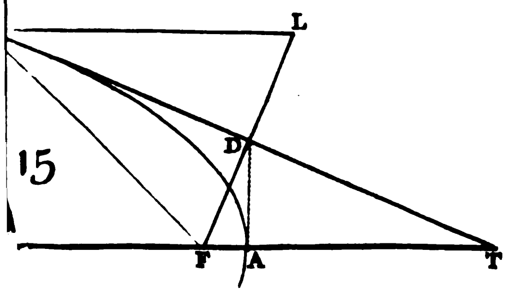

L
R
S
B
5
d
m
q
N
F
P
A
C
a
p
f
M
Q
b
D
n
s
l

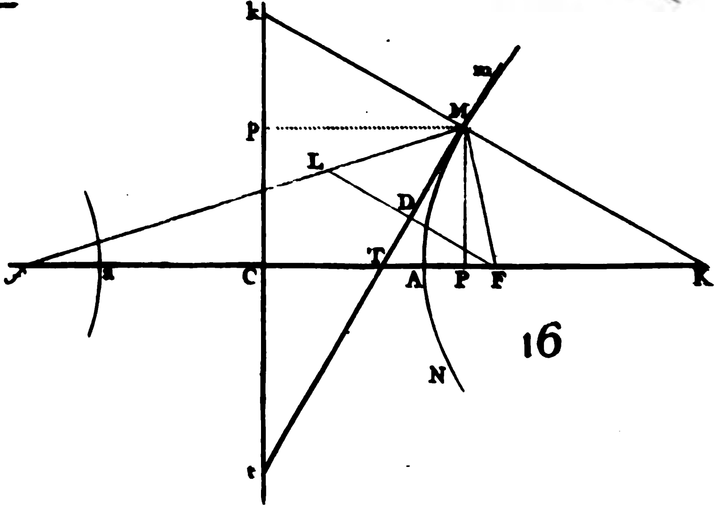

M
M
M
7
f
N
N
F
N
D
M
M
M


E
A
B
L


M
M
M
8
f
D
F
N
N
N
M
M
M


m
B
M
Q
10
f
C
P
a
A
P
q
n
b
N


L
15
D
F
A
T


k
m
p
M
L
D
T
f
a
C
A
P
F
K
16
N
t

UNIV. OF MICH.

In the Ellipſis, if CQ be made $=r-s$, inſtead of being $=s-r$; the demonſtration will ſerve for the caſe, where the ſemi-ordinate MR falls between the center C and the line PD.

When Aa becomes infinite, the proportion will be $\overline{PM}^2 : AP \times 2AC :: p : AC$, or $2p \times AP = \overline{PM}^2$, and the Section * MAN will then be a Parabola. FIG. 30.

Otherwiſe.

From the vertex A of the diameter AP, and from the extremity M of the ſemi-ordinate MP to that diameter, draw the lines AE, MK, perpendicular to the axis BK, and the tangent AT, meeting the axis in T: then if $AE=y$, $BE=x$, $BK=v$, $KM=z$; we ſhall have $2px=yy$, $2pv=zz$, $ET=$ * $2x=\frac{yy}{p}$, $\overline{AT}^2=$ $2px(yy)+4xx$, $EK=u-x=\frac{zz-yy}{2p}$: and becauſe of the ſimilar triangles, $AE:ET::ML(KM-AE):LP=\frac{yz-yy}{p}$; alſo $EK-PL=AP=\frac{\overline{z-y}^2}{2p}$, or $2p \times AP=\overline{ML}^2$: Therefore $\overline{AE}^2 : \overline{AT}^2 :: (\overline{ML}^2)\, 2p \times AP :: \overline{PM}^2 = AP \times \overline{2p+4x}$. But $2p+4x$ expreſſes a given line; conſequently the Squares of the Semi-ordinates to any diameter of a Parabola, are as their correſpondent abſciſſes. * Art. 22.

COROLLARY I.

40. BEcauſe, the 5th, 6th, and 8th Articles are conſequences drawn from the properties of the axes expreſſed by the Equation $\frac{a}{p}yy=2ax\mp xx$, and as that Equation is equally true with regard to any two conjugate diameters; it follows that thoſe conſequences are alſo true with regard to any two conjugate diameters.

COROLLARY II.

41. BEcauſe in the Parabola $\overline{PM}^2 = AP \times \overline{2p+4x}$; it follows, that the parameter $(2p+4x)$ of any diameter AP is equal FIG. 30.

to

* *Art.* 11. to four times the diſtance * of the focus from the vertex of that diameter.

COROLLARY III.

42. IF from the extremities m, N, of the ſemi-ordinates mP, NQ, to the diameter AQ, the lines mH, NF, are drawn parallel to the axis, meeting the tangent in H, F; then will $\overline{AH}^2 : \overline{AF}^2 :: Hm : FN$. For $AH = Pm$, $AF = QN$, and $Hm = AP$, $FN = AQ$; and becauſe $\overline{Pm}^2 : \overline{QN}^2 ::$ * $AP : AQ$; it follows that $\overline{AH}^2 : \overline{AF}^2 :: Hm : FN$. Which ſhews, * *Art.* 41.

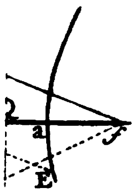
That, *the Lines* Hm, FN, *parallel to the axis terminated by any tangent and by the Curve of a Parabola, are as the Squares of the correſpondent parts of that tangent taken from the point of contact.*

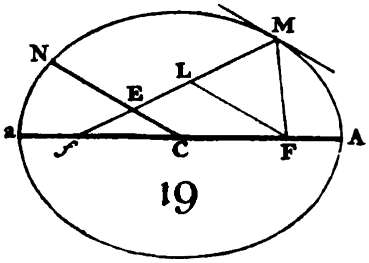
SCHOLIUM.

FIG. 33. I. I*F in the line* AQ *touching the Parabola* AM *in* A, *the parts* Aq, &c. *are taken in an arithmetical progreſſion, the lines* qm, qm, *parallel to the diameter* AP, *terminated by that tangent and the Curve, will be as the Squares* $\overline{Aq}^2$, $\overline{Aq}^2$, *whoſe Sides are in an arithmetical progreſſion; and as the Space* AMQ *may be conceived to contain as many of ſuch lines* qm *with an infinitely ſmall breadth, as there are points in the line* AQ; *it follows that the Space* AmMQ *may be found, in the ſame manner as the ſolidity or content of a Pyramid is found: that is* (*drawing* MD *perpendicular to* MQ *or* AP) by multiplying the greateſt line QM or the baſe, by the third part of the perpendicular MD, which expreſſes their number, *viz*, $MQ \times \frac{1}{3}MD = AmMQ$.

II. *Becauſe the outward Space* AmMQ *is the third part of the circumſcribing Parallelogram* APMQ; *it follows*, that the inward Space AmMP, will be two third parts of the ſame Parallelogram.

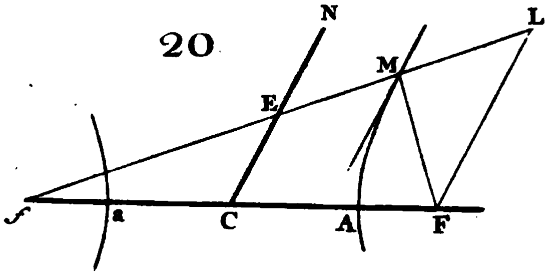
III. *If the parts* Ap, *of the diameter are taken in an arithmetical progreſſion, the Squares made, or the Circles deſcribed by the ſemi-ordinates* pM, pM, *will be in the ſame progreſſion. Therefore,* the Solid generated by the rotation of the Space AMP about the diameter AP, may be found, *in the ſame manner as the Surface of a Triangle is found, that is,* by multiplying the Surface generated by

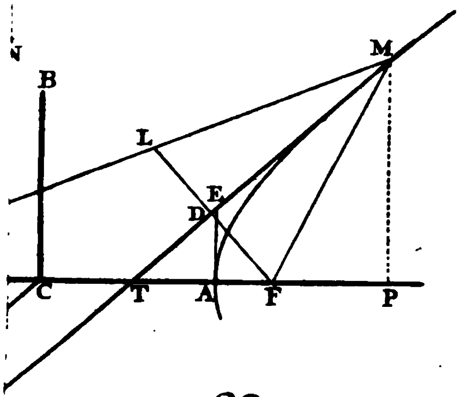




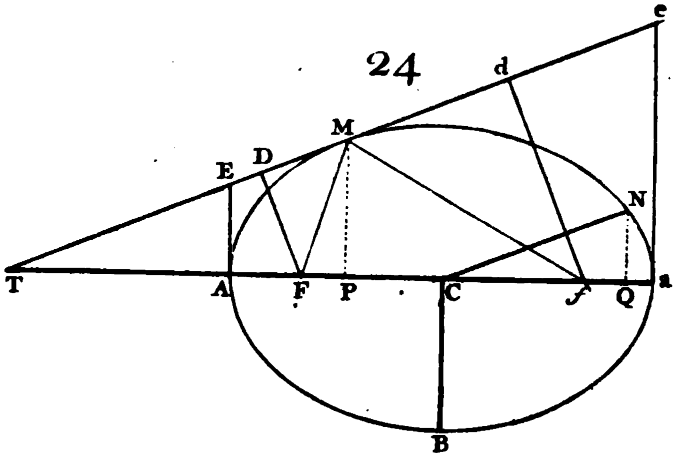
19

20

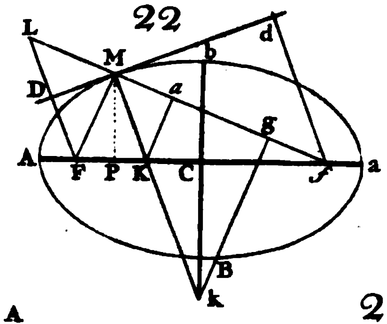




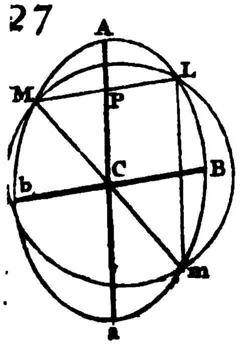
24



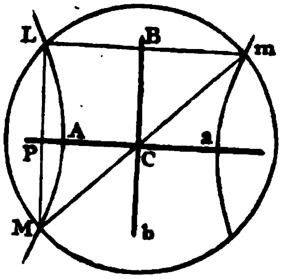
22



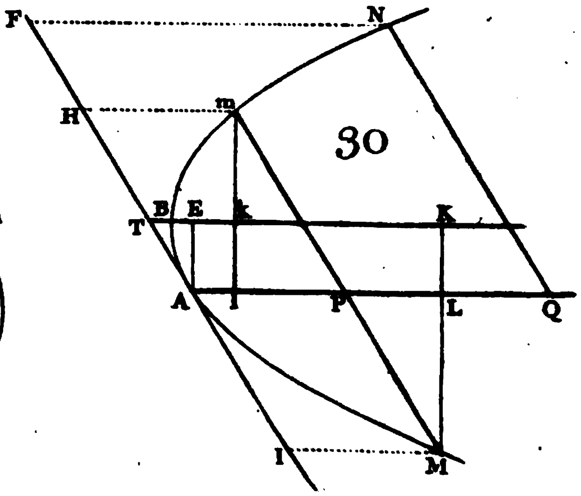
27



28



30



by PM, or the baſe of the Solid, into half the perpendicular height Md.

If Md$=n$, PM$=y$, *and* $c =$ *to the circumference deſcribed by the radius* MD; *then will* $\frac{cy}{2}$ *expreſs the Surface deſcribed by the line* PM, *and* $\frac{cny}{4}$ *will expreſs the Solid required.*

If the Expreſſion $\frac{cny}{4}$ *be taken away from the Expreſſion* $\frac{cny}{2}$ *of the circumſcribing Cylinder, we ſhall have* $\frac{cny}{4}$, *for the expreſſion of the Solid generated by the outward Space* AQM, *revolving about the ſame diameter* AP. *Therefore the two Solids are equal.*

PROBLEM II.

43. *TO deſcribe an Ellipſis, or Hyperbola, when any two conjugate Diameters* Aa, Bb, *are given in poſition and length.* Fig. 34, 35.

Draw a great number of lines MM, mm, &c. parallel to either of the Diameters Aa, and from their interſections P, p, &c. with the other, draw the lines PQ pq, parallel to the line joining two extremities A, B, of thoſe Diameters; then in the Ellipſis, if from the center C, with the radius CA or Ca, you deſcribe the ſemi-circular circumference ADa, and you make always PM $=$ to the correſpondent ſemi-ordinate QN of the Circle: The Curvi-linear Figure AMBamb will be the Ellipſis required.

For the ſimilar triangles CAB, CQP give $CQ = \frac{AC \times CP}{BC}$,

and the property of the Circle $(\overline{QN}^2)$ $\overline{PM}^2 = \overline{AC}^2 - \frac{\overline{AC}^2 \times \overline{CP}^2}{\overline{BC}^2}$,

or $\overline{PM}^2 : \overline{BC}^2 - \overline{CP}^2 :: \overline{AC}^2 : \overline{BC}^2$.

In the Hyperbola, if the line CD be drawn perpendicular and equal to AC, and PM, pm, be made always equal to their correſpondent QD, qD: the curvilinear Figures MAm, Mam, will be the oppoſite Hyperbola's required. For becauſe of the right

G angled,

angled triangle CQD, $(\overline{QD}^2)\ \overline{PM}^2 = \overline{AC}^2(\overline{CD}^2) + \frac{\overline{AC}^2 \times \overline{CP}^2}{\overline{BC}^2}$, or $\overline{PM}^2 : \overline{BC}^2 + \overline{CP}^2 :: \overline{AC}^2 : \overline{BC}^2$.

FIG. 36. In the Parabola, if the parameter AB of the diameter AP, and the angle APM contained by that diameter and its ordinates MM, are given. From the centers C, C, taken at pleaſure in the line BP, and with the radii CB, CB, deſcribe a great number of circular arches BNP, Bnp, and thro' their interſections P, p, with the diameter, draw the lines MM, mm, ſo that the angles MPB, mpB, be equal to the given angle; then if the line An perpendicular to Ab, interſects the circular arch in N, n, and you make always PM equal to its correſpondent AN, the curvilinear Figure mMAMm, will be the Parabola required. For by the property of the circle, $BA \times AP = (\overline{AN}^2)\ \overline{PM}^2$. *W. W. D.*

THEOREM IV.

FIG. 37, 38. 44. THE *Parallelogram* HG, *whoſe ſides touch an Ellipſis or the conjugate Hyperbola's in ſuch a manner, that the points of contact may be joined by any two conjugate diameters* Mm, Nn, *will be always equal to the Rectangle made of the two Axes* Aa, Bb.

From the center C let fall a perpendicular Cr upon one of the ſides, as HM; then becauſe the diameter Nn is parallel to the
* Art. 30. tangent HM, we ſhall have $CN \times Cr = {}^{*}CA \times CB$. Therefore $(4CN \times Cr)\ MNmn = Aa \times Bb$.

THEOREM V.

45. THE *Sum in the Ellipſis, or the Difference in the Hyperbola, of the Squares of any two Semi-conjugate diameters* CM, CN, *is always equal to the Sum or Difference of the Squares of the Semi-axes*, CA, CB.

From the vertex's M, N, of thoſe diameters, draw the ſemi-ordinates MP, NQ to the firſt axis Aa; then will
* Art. 26. $\overline{CM}^2 = yy + uu$, and $\overline{CN}^2 = zz + (vv =)^{*} \pm aa \mp uu$. Therefore
* Art. 27. $\overline{CN}^2 \pm \overline{CM}^2 = bb \pm aa$, becauſe $zz \pm yy = {}^{*}bb$.

THE-

THEOREM VI.

46. I*N the Ellipſis or Hyperbola, if any right line* MN *interſects a diameter* Dd *within or without the Curve: the Rectangle of its parts* ML, LN, *terminated by the Curve, and that interſection* L, *will be to the Rectangle of the parts* DL, Ld, *of the diameter, terminated likewiſe by the Curve, and the interſection* L, *in a given ratio, that is* (*drawing the ſemi-diameter* BC *parallel to the line* MN) $MLN : DLd :: \overline{BC}^2 : \overline{DC}^2$. FIG. 39, 40.

Through the middle P of MN, draw the diameter Aa, and from the vertex D of the diameter Dd, the ſemi-ordinate Dp to Aa. Then if $Cp=v$, $pD=z$, we ſhall have * $yy=\pm ap\mp\frac{puu}{a}$, $zz=\pm ap\mp\frac{pvv}{a}$. And becauſe of the ſimilar triangles * Art. 39.

CpD, CPL; PL will be $=\frac{uz}{v}$. Alſo $yy-\frac{uu}{vv}zz=MLN$, when the point L falls within the Curve; or ſubſtituting the * * Art. 2. values of yy, zz, $\pm bb\mp\frac{bbuu}{vv}=MLN$, becauſe * $ap=bb$.

Therefore $MLN:\overline{BC}^2 :: \pm\overline{Cp}^2\mp\overline{CP}^2:\overline{Cp}^2 :: \pm\overline{CD}^2\mp CL = DLd : \overline{DC}^2$, or $MLN:DLd::\overline{BC}^2:\overline{DC}^2$. *W. W. D.*

By changing all the Signs, the Proportion will ſerve for the caſe, when the point L falls without the curve.

If Ld becomes infinite, and alſo = * Dd, the Proportion will become $MLN:DL\times 2DC::\overline{BC}^2:\overline{DC}^2$, or $MLN:2DL::\overline{BC}^2:DC$; and the Section will then be a Parabola. * *By the Poſtulat.*

Otherwiſe.

Becauſe * $2p\times AP=\overline{PM}^2$, and $2p\times Ap=\overline{PD}^2=\overline{PL}^2$; we ſhall * Art. 39. have $(\pm\overline{AP\mp Ap}\times 2p=\pm\overline{PM}^2\mp\overline{PL}^2)$ or $2p\times DL=MLN$. FIG. 41. Therefore $MLN:DL::2p:1$.

COR.

COROLLARY I.

Fig. 39, 40. 47. HEnce, if the line mn parallel to the ſemi-diameter RC, interſects the diameter Dd in the ſame point L, as the line MN, and is terminated by the curve in m,n; then will MLN : $\overline{BC}^2$ :: mLn : $\overline{RC}^2$ :: DLd : $\overline{DC}^2$, in the Ellipſis, and Hyperbola. And if *q* * be the parameter of the diameter belonging to the ordinate mn; then will MLN:mLn::p:q::DL:1, in the Parabola.

* Fig. 41.

COROLLARY II.

F. 39, 40, 41. 48. IN any Conic-Section, if the line H,K, parallel to the line mn, interſects the curve in H,K, and the line MN in E; then will MLN:mLn :: MEN:HEK. For, becauſe MLN : mLn :: * $\overline{BC}^2$: $\overline{RC}^2$, or :: *p* : *q*; when the line mn ſalls upon the line HK, the line Lm will become = EH, Ln = EK, ML = ME, and LN = EN: Therefore MEN : HEK :: $\overline{BC}^2$: $\overline{RC}^2$, or :: *p*: *q*. Conſequently MLN : mLn :: MEN : HEK.

* *Art.* 47.

COROLLARY III.

Fig. 42. 49. IN any Conic-Section, if the line DP (the reſt as before) parallel to the lines mn, HK, touches the curve in D, and meets the line MN produced in P; then will MLN : mLn :: NPM : $\overline{DP}^2$. For if the line HK be conceived to fall upon the tangent DP; EH will become = EK = DP, EN = PN, and EM = PM. Therefore the proportion * MLN : mLn :: MEN : HEK will then become MLN : mLn :: NPM : $\overline{DP}^2$.

* *Art.* 48.

50. Hence it follows 1.° If the line BO parallel to the line NM, touches the curve in B, and meets the tangent DP in O; by conceiving the line NM to fall upon the tangent OB; then will MLN : mLn :: $\overline{BO}^2$: $\overline{DO}^2$.

2.° If the line AR parallel to the tangent DO, touches the Curve in A, and meets the tangent OB in R; by conceiving the lines HK, MN, to fall upon thoſe tangents; then will $\overline{BO}^2 : \overline{DO}^2 :: \overline{BR}^2 : \overline{AR}^2$, or BO : DO :: BR : AR.

3.° If the lines nm, KH produced, meet the tangent OR in G, F; we ſhall have 1.° HFK : mGn :: $\overline{BF}^2 : \overline{GB}^2$. 2.° $\overline{OB}^2 : \overline{DO}^2$ :: NEM : HEK. 3.° mGn : $\overline{GB}^2 :: \overline{DO}^2 : \overline{OB}^2$.

THEOREM VII.

51. *IN any Conic-Section, if from a point* T *taken in a diameter* Aa, *produced if neceſſary, you draw a line* Tm, *interſecting the Curve in two points* H, m, *and from thoſe points the ordinates* HK, mn, *to that diameter: I ſay the line paſſing thro' the other extremities* n, K, *of thoſe ordinates, will meet the diameter in the ſame point* T. Fig. 43, 44.

For let thoſe ordinates meet the diameter in p, Q; then will Qm = Qn, pH = pK; and becauſe of the parallel lines mn, HK, we ſhall have Qm — pH : Qp :: pH : pT, and Qn — pK : Qp :: pK : pT. Therefore, ſince the three firſt correſpondent terms of the two proportions are equal, the fourths will alſo be equal. *W. W. D.*

COROLLARY I.

52. HEnce, when the line Tm becomes the tangent TM; the line Tn will likewiſe become the tangent TM; and the line joining the points of contact M, M, will be parallel to the lines mn, HK.

THEOREM VIII.

53. I *Say the parts* TH, Tm, *of a line drawn from the point of concourſe* T *of two tangents to any Conic-Section, terminated by the Curve and that point, are always to each other as the parts* rH, rm, *of the ſame line terminated likewiſe by the Curve and by the line joining the points of contact, viz.* TH : Tm :: rH : rm.

Let the lines nm, KH produced, meet the tangent TM in G, F. Then becauſe of the parallel lines Gn, MM, FK, we ſhall have TF : TG :: HF : mG :: FK : Gn, or $\overline{TF}^2 : \overline{TG}^2$ ::
* Art. 50. No. 3. HFK : mGn :: * $\overline{MF}^2 : \overline{MG}^2$. Therefore TF : TG :: MF : MG; or TH : Tm :: rH : rm. *W. W. D.*

COROLLARY I.

54. WHen the line Tm falls upon the diameter Ta; then will TH become = TA, Tr = TP, and Tm = Ta. Therefore TA : Ta :: PA : Pa. If CT = n, the proportion will give, $n-a : n+a :: a-u : a+u$, or $aa = nu$, that is, ∺ CP : CA : CT, in the Ellipſis and Hyperbola, and TA = AP in the Parabola; becauſe Ta and Pa become infinite, and alſo equal in that caſe.

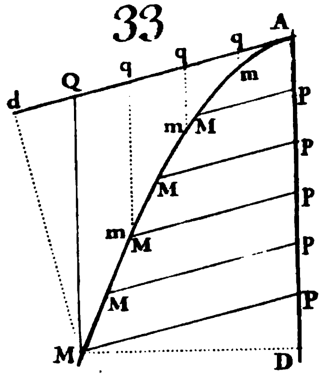
COROLLARY II.

55. FRom whence may be deduced in the ſame manner as in the Article 20, (retaining the ſame nomination) the following expreſſions: PT = $\mp\frac{aa \mp uu}{u}$, AT = $\mp\frac{aa \mp au}{u}$, TP : PM :: $y : \frac{pu}{a}$, $uu = aa \mp vv$, $zz \mp yy = bb$, $\frac{ay}{b} = v$, $\frac{az}{b} = u$, $uy = vz$, and PCT = APT.

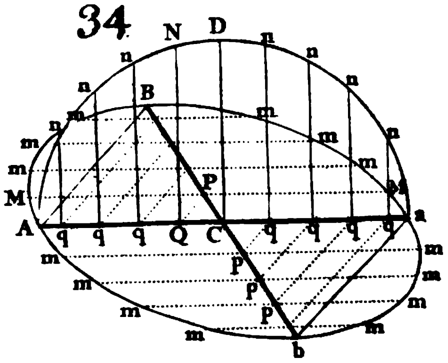
COROLLARY III.

56. BEcauſe TH : Tm :: Hp (pK) : mQ and Hr : rm :: Pp : PQ; it follows that pK : mQ :: Pp : PQ,
* Art. 53. ſince * TH : Tm :: Hr : rm. Therefore the adjacent ſides of the equal angles mQP, KpP being proportional, the triangles mQP, KpP will be ſimilar; and as the ſides QP, Pp, are in a right line; the ſides mP, PK, will be likewiſe in a right line.

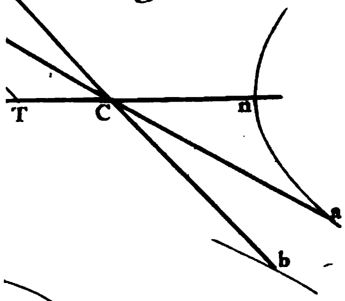
33



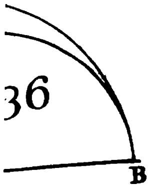
34



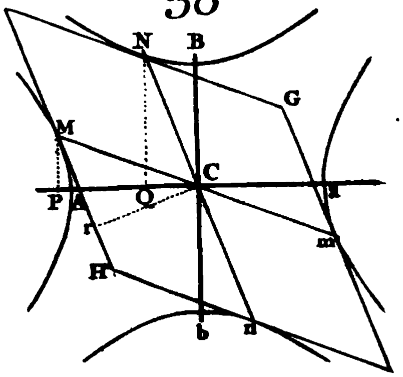
32



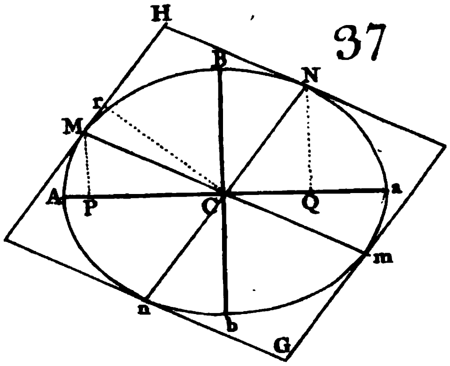
36

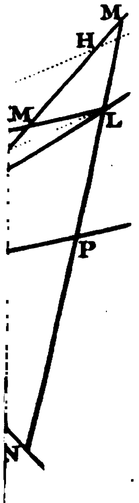


38

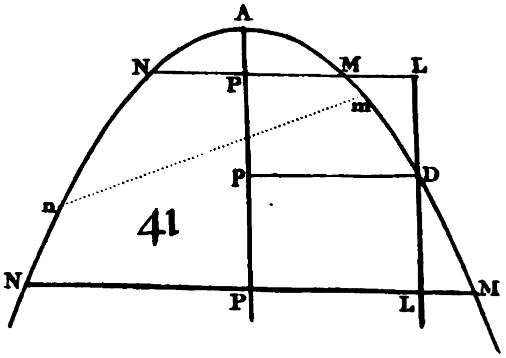


37

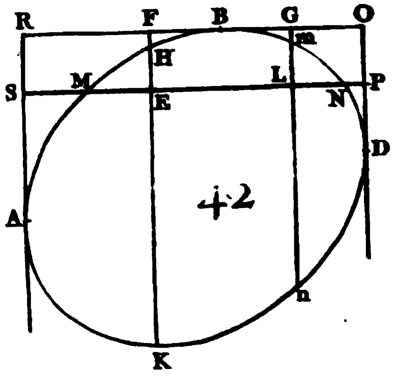




41



42



UNIV. OF MICH.

COROLLARY IV.

57. IF thro' the point T, be drawn a line TV parallel to the ordinates MM of the diameter Aa, and the line mK, produced if neceſſary, meets TV in V: it is evident that the line mV, is divided in the points V, K, P, m, in the ſame proportion, as the line Tm, is in the points T, H, r, m, *viz.* VK : Vm :: PK : Pm.

COROLLARY V.

58. HEnce the parts VK, Vm, of any line drawn thro' the point P, intercepted between the Curve and the line TV, are always to each other, as the parts PK, Pm, intercepted between the Curve and the point P. For it is plain, (by ſuppoſing the lines mV, mT, to revolve about the points P, T) that whilſt the interſection m paſſes over the arch MaM, nam, the interſection V will paſs over all the points of the line TV.

COROLLARY VI.

59. IF from any point D of the line TV, (the reſt as before) there be drawn two tangents Dm, DK, to the Section; Fig. 45. the line mK joining their points of contact, will always paſs thro' the given point P. For, conceiving a line drawn thro' the points D, P; it is plain that its parts terminated by the point D and the Curve, are to each other, as thoſe terminated by the point P, and the Curve. Therefore the line mK will paſs thro' the * point P. * Art. 53.

COROLLARY VII.

60. THE line Hd touching the Section in H, will meet the line PM produced, in the ſame point t, as the tangent mD. For becauſe of the parallel lines tM, HK, TD, the lines TD, Td, will be equal, ſince pH=pK; and TH : Hr :: Td : rt; likewiſe Tm : rm :: TD : rt. But * TH : Hr :: Tm : rm; therefore Td : rt :: TD : rt. Conſequently, ſince the antecedents Td, TD, are equal, the conſequents are alſo equal. From whence we deduce the following, * Art. 53.

THE

THEOREM IX.

61. IF upon a Diameter Aa of any Conic-Section, produced if neceſſary, there be taken the points P, T, ſuch that ∺ $CP : CA : CT$, in the Ellipſis or Hyperbola, and $AP = AT$ in the Parabola, and the lines Pt, TV, are drawn parallel to the ordinates of that Diameter; the line joining the points of contact of two tangents, drawn from any point of the line DT to the Section, will always paſs thro' the given point P; and the line joining the points of contact of two tangents, drawn from any point of the line Pt, will always paſs thro' the given point T.

THEOREM X.

Fig. 46, 47. 62. THE *Tangents drawn from the points* T, t, *taken in any Diameter of an Ellipſis or Hyperbola, equally diſtant from the center* C, *will form a Parallelogram* DtdT.

Through the points of contact draw the ordinates Mm, Nn. Then becauſe * ∺ $CP : CA : CT$, and ∺ $CQ : Ca : Ct$, and CT being $= Ct$, $CA = Ca$; the abſciſs CP will be equal to the abſciſs CQ. Therefore $PM = Qn$, and $PT = Qt$. But becauſe the ſides TP, PM, and tQ Qn, adjacent to the equal angles TPM, tQn, are equal; the triangles TPM, tQn, will be ſimilar and equal; and as the ſides TP, tQ, are in a right line, the ſides TM, tn, will be parallel. In the ſame manner will be proved, that the ſide Tm is parallel to the ſide tN. *W. W. D.*

* *Art.* 54.

COROLLARY I.

63. THE parts DM, dn, or TM, tn, between the points of contact, and the oppoſite angles of a Parallelogram, whoſe ſides touch an Ellipſis, or the oppoſite Hyperbola's, are equal. For $TD = td$, and $TM = tn$. Therefore $DM = dn$.

COROLLARY II.

64. THE Diagonals Tt, Dd, of a Parallelogram, whoſe ſides touch an Ellipſis, or the oppoſite Hyperbola's,

will be two conjugate Diameters. For becauſe MP=NQ, the line NM will be parallel to QP, and the triangles NDM, tDT, will be ſimilar; therefore, ſince DC biſects tT, it will likewiſe biſect NM, as well as nm; and conſequently will be the conjugate diameter to Aa.

THEOREM XI.

65. *If a line* LS *interſects two tangents* TA, TC, *to any* F. 48,49,50. *Conic-Section; I ſay, the Squares of its parts, terminated by thoſe tangents, and the line* AC *joining the points of contact will be as the Rectangles of the parts of the ſame line, terminated by the Curve and the tangents,* viz. $\overline{SN}^2 : \overline{SL}^2 :: RNH : HLR$.

From the point L draw the line LF parallel to the tangent TC, meeting the Curve in E, F, and the line AC in K; then will ELF : *$\overline{LA}^2 :: \overline{TC}^2 : \overline{TA}^2$. And becauſe the lines LK, * *Art.* 50. *No.* 3. TC, are parallel, we ſhall have $\overline{LK}^2 : \overline{LA}^2 :: \overline{TC}^2 : \overline{TA}^2$; alſo $\overline{LK}^2 = ELF$. Therefore ($\overline{CN}^2 : \overline{LK}^2 = ELF$ ::) $\overline{SN}^2 : \overline{SL}^2$ :: * * *Art.* 50. RNH : HLR. *W. W. D.*

N. B. When the line LR becomes a Diameter in the Parabola, Fig. 49. it will meet the Curve only in one point H; therefore NR, LR, will become infinite and equal in this caſe. Conſequently the former proportion will become $\overline{SN}^2 : \overline{SL}^2$ :: RN × NH : HL × RN : NH : HL.

COROLLARY I.

66. If the line LS, inſtead of interſecting the Curve, only touches it in B, that is, if the line LS becomes the tangent lS; Fig. 50. then will LH become = LR = lB, NR = NH = nB, and SH = SR = SB. Therefore $\overline{Sn}^2 : \overline{Sl}^2 :: \overline{Bn}^2 : \overline{Bl}^2$, or Sn : Sl :: Bn : Bl.

COROLLARY II.

67. The Diagonals MG, LN of a Trapezium MNGL, whoſe ſides touch any Conic-Section, will interſect each Fig. 51, 52.

other in the ſame point S, as the lines AC, BD, joining the oppoſite points of contact. For let the diagonal LN interſect the Curve in H, R; then the ſquares of the parts SL, SN, of any line as LN, terminated by two tangents LA, NC, and by the
* *Art.* 65. line AC joining their points of contact, will be as the * rectangles HLR, RNH, that is, RNH : HLR :: $\overline{SN}^2 : \overline{SL}^2$, or $\sqrt{RNH} : \sqrt{HLR}$ :: SN : SL; and by compoſition $\sqrt{RNH} + \sqrt{HLR} : \sqrt{HLR}$ :: LN : SL. But the ſame proportion will hold good with reſpect to the tangents LB, ND, and the line BD joining their points of contact: Therefore, ſince the three firſt terms are invariable, the fourth muſt neceſſarily remain the ſame. The ſame thing will be proved with reſpect to the diagonal MG.

COROLLARY III.

68. IF the lines, joining the points of contact B, C and A, D, are produced, they will meet the diagonal LN, likewiſe produced if neceſſary, in the ſame point E. For we ſhall have
* *Art.* 65. * $\overline{EN}^2 : \overline{EL}^2$ :: RNH : HLR, for both the lines BC, and AD. And becauſe RNH : HLR :: $\overline{SN}^2 : \overline{SL}^2 :: \overline{EN}^2 : \overline{EL}^2$; it follows that SN : SL :: EN : EL.

N. B. The line KT, joining the two interſections of the four
FIG. 52. tangents falling without the Section, will likewiſe interſect the diagonal NL, in the ſame point E, as the lines DA, CB pro-
* *Art.* 53, 60. duced. For we have proved, * that if four tangents AM, MD, BG, GC, interſect two by two each other in a right line MG; the lines DA, CB, joining their points of contact, are divided by their interſection E, and by the line MG, ſuch that EA : ED :: An : nD, as alſo any line drawn from the point E, that is, EH : ER :: SH : SR. But this is likewiſe the property of
* *Art.* 61. the * line KT: therefore &c.

THEOREM XII.

69. IF *two parallel lines* NM, LE, *are terminated by the Curve*
FIG. 53. *of a Parabola; the Rectangles of their parts, terminated by the interſections of any two diameters* bD, mF, *are as the parts of thoſe*

thoſe Diameters, intercepted between their vertex's b, m, *and thoſe lines, viz.* mF : bD :: MFN : EDL.

Let p be the parameter of the diameter AP, belonging to the ordinates MN, EL; then will $p\times$mF =* MFN, and $p\times$bD =EDL. Therefore $p\times$mF : $p\times$bD :: mF : bD :: MFN : EDL. *W.W.D.* * *Art.* 46

Corollary I.

70. If the line AH parallel to the line NM, touching the Parabola in A, meets the diameters bD, mF produced, in R, Q; then by conceiving the line LE to coincide with that tangent, we ſhall have mF : bR :: MFN : $\overline{AR}^2$. And conceiving the line NM to fall upon AH; then will bD : mQ :: LDE : $\overline{AQ}^2$.

Corollary II.

71. If the diameter bD interſects the line NM in K; by conceiving the line LE to fall upon NM, we ſhall have NKM : NFM :: bK : mF, and NKM : LDE :: bK : bD.

Corollary III.

72. Hence a Parabola may be deſcribed thro' three given points N, m, M, when the poſition of the diameters is given.

For if you draw thro' one of thoſe points as m, and thro' the middle P of the line NM joining the other two, the diameters AP, mF, and you take MFN : MPN or $\overline{PM}^2$ :: mF : AP; the vertex A of the diameter AP will be given, and a third proportional to AP and PM will be its parameter. Therefore the Parabola may be deſcribed as in the Article 43.

Corollary IV.

73. If two lines DR, MN, terminating in a Parabola, interſect each other in L, and are parallel to two lines EB, AB, touching the Curve in E, A; then drawing thro' the point of Fig. 54.

of contact A, the Diameter AP meeting BE in T, and from the point N, the line Nb parallel to AP, meeting likewiſe RD
* *Art.* 54. produced in b: we ſhall have $\overline{EB}^2$ * or $\overline{TB}^2 : \overline{AB}^2$ :: * RLD :
* *Art.* 50. MNL :: $\overline{Lb}^2 : \overline{LN}^2$, or RLD : ML :: $\overline{Lb}^2$: LN. And if NQ parallel to RM, meets RD in Q, then will ML : RL :: LN : LQ, *ex aequo*, LD : 1 :: $\overline{Lb}^2$: LQ or LD×LQ = $\overline{Lb}^2$.

COROLLARY V.

74. HEnce a Parabola may be deſcribed thro' four given points, R, M, D, N. For, retaining the foregoing conſtruction, if Lb be made a mean proportional between LD and LQ: the vertex A of the diameter AP, drawn thro' the
* *Art.* 72. middle P of the line MN, parallel to Nb, may be found * as well as its parameter; and therefore the Parabola may be deſcribed.

N.B. When the point b can be taken on both ſides of the point L, there may be deſcribed two different Parabola's thro' the four given points: And when both the points b fall into the points R, D, the conſtruction is impoſſible; becauſe the diameter Nb would then paſs thro' two points of the Curve, which is impoſſible.

DEFINITIONS.

14.

Fig. 55. THE Lines TL, RV, drawn thro' the center C of an Hyperbola, parallel to the lines BA, bA, joining the vertex's of the axes Aa, Bb, are called *Aſymptotes.*

15.

The Square of the part CD or CH of the Aſymptote, terminated by the line BA, or bA, is call'd the *Power* of the Hyperbola.

COROLLARY I.

75. HEnce the tangent EF drawn thro' any of the vertex's A of the axes, and terminated by the Aſymptotes, is

equal to the axis to which it is parallel. For becauſe of the parallel lines BA, CH, and bA, CD; CB will be = EA, and bC = AF.

Corollary II.

76. THE Power $\overline{CD}^2$ of the Hyperbola is $= \frac{\overline{CA}^2+\overline{CB}^2}{4}$. For becauſe of the parallel lines BA, CH, the line CD is = AH = ½ bA, or 2CD = bA = AB; and becauſe of the right angled triangle ACB, $(\overline{AB}^2)\ 4\overline{CD}^2 = \overline{CA}^2 + \overline{CB}^2$, or $\overline{CD}^2 = \frac{\overline{CA}^2+\overline{CB}^2}{4}$.

THEOREM XII.

77. *IF a line* MT *be drawn parallel to either of the axes; I ſay the Rectangle of its parts* MR, MT, *terminated by the aſymptotes and by any of the Hyperbolical Curves, is equal to the Square of the ſemi-axis* CB, *to which it is parallel, viz.* $RMT = \overline{BC}^2$.

For becauſe of the ſimilar triangles CAE, CPT, we have $CA\,(a) : AE\,(b) :: CP\,(u) : PT = \frac{bu}{a}$. Alſo $TM = TP + PM = \frac{bu}{a} + y$, $MR = \pm PT \mp PM = \pm \frac{bu}{a} \mp y$. Therefore $RMT = \pm \frac{bbuu}{aa} \mp yy = {}^{*}\, bb$. *W. W. D.*

* *Art.* 39.

Corollary I.

78. HEnce, if there be drawn two right lines rt, *a*d, parallel to each other, the Rectangles rMt, *a*Nd, of their parts terminated by the aſymptotes and by any of he Hyperbolical Curves, will be equal. For if the lines drawn thro' their interſections M, N with the Curves, parallel to either of the axes, meet the aſymptotes in R, T, L, V; then will MR :

Mr :: NV : Nd, and MT : Mt :: NL : N*a*; and multiplying orderly, RMT : rMt :: LNV : *a*Nd. But RMT = *LNV. Therefore rMt = *a*Nd.

* *Art.* 77.

Corollary II.

79. If from any two points M, N, taken in the Hyperbolical Curves, the lines M*a*, Nr and Md, Nt, are drawn parallel to each other, meeting the aſymptotes in *a*, r, d, t; the Rectangle *a*Md of thoſe drawn from one point, will be equal to the Rectangle rNt of thoſe drawn from the other. For let the lines drawn thro' the points M, N, parallel to each other, meet the aſymptotes in R, T, V, L; then will MR : M*a* :: NL : Nr, and MT : Md :: NV : Nt; and multiplying orderly, RMT : *a*Md :: LNV : rNt. But RMT = *LNV. Therefore *a*Md = rNt.

Fig. 56. * *Art.* 78.

Corollary III.

80. If the lines M*a*, Nr, are parallel to the aſymptote TV, and Md, Nt, parallel to the aſymptote RL; then will C*a* = dM, and Cr = tN. Therefore C*a*M = CrN.

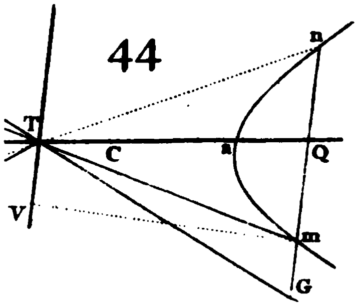
If the line Kg joining the vertex's of the axes, interſects the aſymptote RL in I, and CI = IK = a, C*a* = x, *a*M = y; then by conceiving the line rN to coincide with the line IK, we ſhall have $aa = xy$, for the Equation of the Hyperbola with reſpect to the Aſymptotes.

Corollary IV.

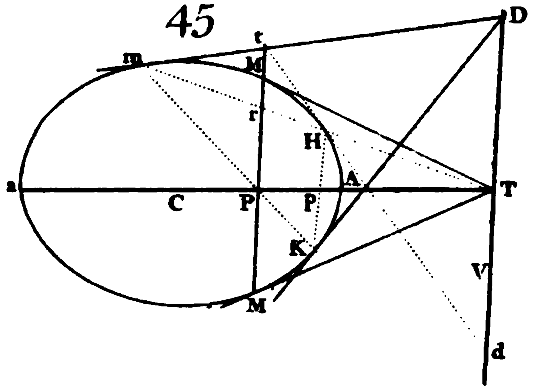
81. Any Tangent EF to the Hyperbola, terminated by the aſymptotes, is biſected in its point of contact A, and is equal to the diameter Bb, to which it is parallel. For let the lines ST, VL, parallel to EF, meet the aſymptotes in R, T, V, L, and the Curves in S, X, N, n. Then becauſe RST = *TXR = LNV = VnL, when the line ST coincides with the diameter Bb, and VL with the tangent FE; the Equation RST = TXR = LNV = VnL, will become $\overline{BC}^{2} = \overline{bC}^{2} = \overline{EA}^{2} = \overline{AF}^{2}$, or BC = EA = AF.

* *Art.* 78.

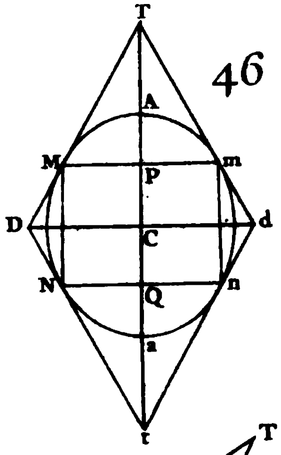
44

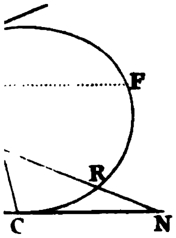


45

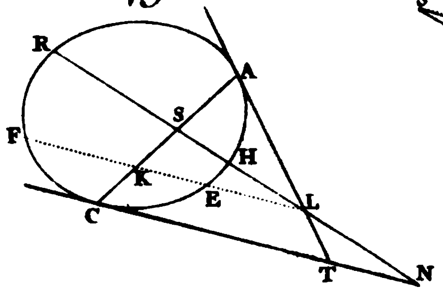


46

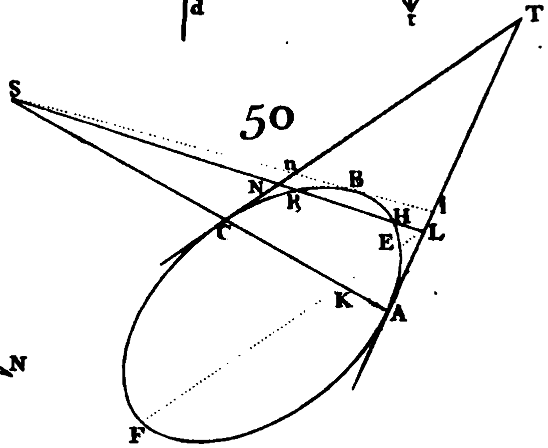


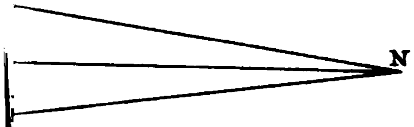


49

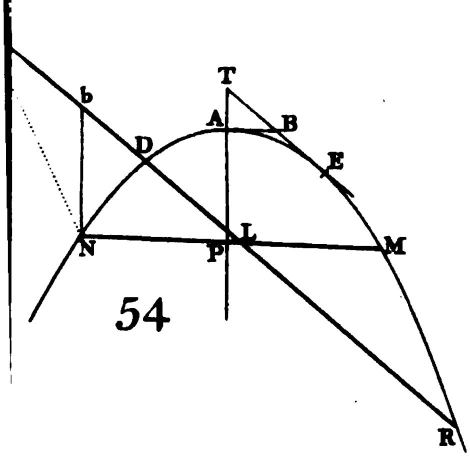


50

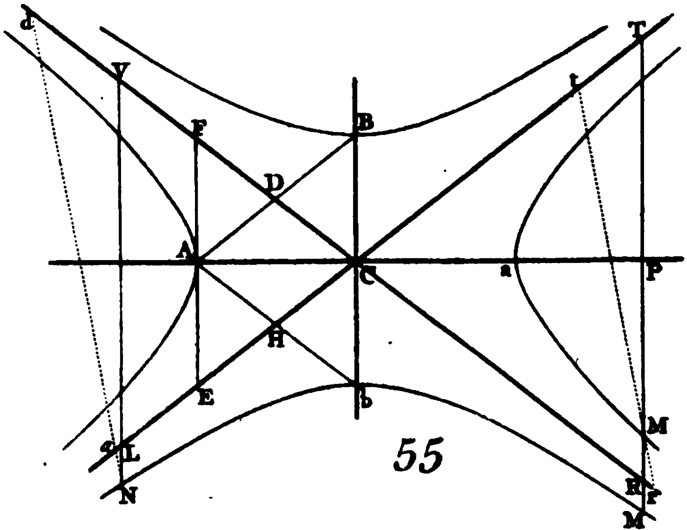




54



55



COROLLARY V.

82. THE parts SR, TX or Vn, LN of any right lines, terminated by the aſymptotes and the Hyperbolical Curve, or by the oppoſite Curves, are equal. For becauſe * Art. 78. $LNV^{*}=VnL$, or $LN\times\overline{Nn+nV}=nV\times\overline{Nn+NL}$; we ſhall have $LN\times Nn=nV\times Nn$, or $LN=nV$. In the ſame manner will be proved that $SR=TX$.

COROLLARY VI.

83. THE lines AB, Ab, joining the vertex's of any two Conjugate diameters Aa, Bb, are parallel to, and biſected by the aſymptotes. For becauſe BC is parallel and $=^{*}AE$, the line BA will be parallel to the aſymptote CH, * Art. 81. which therefore biſects the line Ab, ſince $BC=Cb$: and becauſe the line AF is parallel and equal to bC, the line bA will be parallel to the aſymptote CD, which biſects alſo the line AB.

COROLLARY VII.

84. THE Aſymptotes of an Hyperbola being given, there may be drawn a line EF, ſo as to touch the Hyperbola in a given point A. For if the line AD be drawn parallel to the aſymptote CH, and DF be taken $=DC$; the line drawn thro' the points A, F, will be the tangent required: Since AE being $=^{*}AF$; CD will be $=DF$. * Art. 81.

COROLLARY VIII.

85. ANY two conjugate Diameters Aa, Bb, of an Hyperbola being given, the axes Kk, Gg, may be found. For drawing the aſymptote CE thro' the middle H of the line Ab, and gK parallel to bA, ſo that $CI=IK=Ig=^{*}\sqrt{CH\times HA}$; * Art. 80. the lines drawn thro' the center and the points g, K, will be the axes required.

COROLLARY IX.

86. BEcauſe the adjacent ſides CH, HA, and CI, IK of
* *Art.* 80. the equal angles H, I, are reciprocally * proportional, it follows, 1.° That the triangles CAH, CKI, are equal; and therefore the Parallelogram, whoſe ſides touch the conjugate Hyperbola's, and are parallel to any two conjugate diameters, $=4CEF=4bCA$, is always given. 2.° That the difference of the Squares $\overline{CA}^2-\overline{Cb}^2$ of any two conjugate diameters (propor-
* *Art.* 12. tional to $^{*}4CH\times HA$) is always given. This has been proved already in the Articles 44, 45.

SCHOLIUM.

Fig. 57. I*N the Hyperbola, if the lines* PM, PM, *&c. are ſemi-ordinates to the diameter* CA *conjugate to* BC, *the line* AE *a tangent in* A, CN *an aſymptote, and* Bb *a line parallel to* CA, *interſecting the ſemi-ordinates in* b, b, *&c*; *and if the Figure* CBMP *revolves*
* *Art.* 78. *about the diameter* CA. *Then, becauſe* $(NM\times\overline{NP+PM}={}^{*}\overline{AE}^2)$ $\pm\overline{PN}^2\mp\overline{PM}^2=\overline{AE}^2=\overline{CB}^2$, or $\overline{PM}^2=\overline{PN}^2\mp\overline{BC}^2$, *the Surface deſcribed by* PM, *will be always equal to the Surface deſcribed by* PN, *plus or minus, the Surface deſcribed by* $Pb=CB$. *And becauſe this happens every where, the* Solid *generated by the ſpace* AMP, *will be equal to the Difference of the* Fruſtum *of a Cone generated by the trapezium* AENP, *and the* Cylinder *generated by the parallelogram* AEbP.

Aud the Solid *generated by the Space* CBMP, *will be equal to the Sum of the* Cone *generated by the triangle* CPN, *and the* Cylinder *generated by the parallelogram* CBbP.

THEOREM XIII.

87. I*N the Hyperbola, if there be drawn a line* BD *parallel to*
Fig. 88, 59. *one of the Aſymptotes* CT; *I ſay its Part* BP *terminated by the Curve and the interſection* P *of any other line* MN *given in poſition, will be to the Rectangle* MPN *of the parts of that other line, terminated likewiſe by the Curve and the interſection* P, *in a given ratio.*

Let the line MN, produced if neceſſary, meet the aſymptote in Q, and the diameter CA, to which it is an ordinate, in L, and from the point B draw the line BT parallel to the line NM, meeting the aſymptote in T, and the diameter in K. Then if the tangent AE, drawn thro' the vertex A, meets the aſmptote in E, and CE $= a$, AE $= b$, BD $= c$, PD $= x$, PM $= y$, BT or PQ $= r$; the ſimilar triangles CAE, DLP, DKB, will give PL $= \frac{bx}{a}$, BK $= \frac{bc}{a}$. Therefore * QL $= r \mp \frac{bx}{a}$, ML $= y \mp \frac{bx}{a}$, TK $= r \mp \frac{bc}{a}$. But $(\overline{TK}^2 - \overline{KB}^2 =$ * $\overline{QL}^2 - \overline{ML}^2) \mp \frac{2bcr}{a} = \mp \frac{2brx}{a} \pm \frac{2bxy}{a} - yy$, or $yy \mp \frac{2bxy}{a} = \frac{2br}{a} \times \pm \overline{c \mp x}$. Conſequently 2BP×PQ $(\pm 2cr \mp 2rx)$: MPN $(yy \mp \frac{2bxy}{a})$ :: CE (a) : AE (b), or BP : MPN :: CE : 2PQ×AE.

* Fig. 58.

* Art. 78.

By changing all the Signs, the Demonſtration will ſerve for the Figure 59.

COROLLARY I.

88. HEnce, if a line mn parallel to the line MN, interſects the Curve in m, n, the line BP in p, and the aſymptote in q; we ſhall have 2BP×PQ : MPN :: 2Bp×pq : mpn :: $a : b$. And when the line * mn becomes the tangent AF; then will Bp become $=$ BF, pm $=$ pn $=$ AF. Therefore 2BP×PQ : MPN :: 2BF×FE : $\overline{AF}^2$.

Fig. 58, 59.

* Fig. 58.

THEOREM XIV.

89. I*F a line* LL, *terminated by the Curve of an Ellipſis or Hyperbola, interſects any two conjugate diameters* Mm, Nn, *without or within the Section; I ſay the Rectangle of its parts, terminated by thoſe interſections and by the interſection of the diameter* Aa, *to which it is an ordinate,* plus *or* minus *the Square of half*

Fig. 60, 61.

half that line, will be equal to the Square of the ſemi-diameter BC *drawn parallel to it, viz.* $a\text{Rd} \pm \overline{\text{RL}}^2 = \overline{\text{BC}}^2$.

Thro' the vertex's M, N, draw the ſemi-ordinates MP, NQ, to the diameter Aa, and make CQ $= v$, QN $= z$; then becauſe of the ſimilar triangles CQN, CRd, and CPM, CRa, we ſhall have $a\text{R} = \frac{y}{u}\,\text{CR}$, $\text{dR} = \frac{z}{v}\,\text{CR}$, and $a\text{Rd} = \frac{bb}{aa}\,\overline{\text{CR}}^2$
* Art. 55. (ſince $\frac{a}{b}\,y = {}^{*}v$, $\frac{az}{b} = u$). Therefore $\frac{bb}{aa}\,\overline{\text{CR}}^2 \pm \overline{\text{RL}}^2 = a\text{Rd} \pm$
* Art. 39. $\overline{\text{RL}}^2 = {}^{*}\overline{\text{BC}}^2$. *W.W.D.*

When the line Ra becomes the tangent AE; then will RL $= 0$, Rd $=$ AF, and $a\text{Rd} \pm \overline{\text{RL}}^2$ will become FAE $= \overline{\text{BC}}^2$.

And when AE $=$ AF in the Ellipſis; then will Ra become $=$ Rd, and $\overline{a\text{R}}^2 + \overline{\text{RL}}^2 = \overline{\text{BC}}^2 = \overline{\text{FA}}^2$.

SCHOLIUM.

Fig. 62. I*F* ABC *be an Elliptical Quadrant,* CA, CB, *any two ſemi-conjugate diameters,* AF *parallel and equal to* CB, *and if the ſemi-ordinate* RL *to* AC, *meets in* b *the line joining the points* B, F,
* Art. 89. *and the Diameter* CF *in a. Then becauſe* $\overline{a\text{R}}^2 + \overline{\text{RL}}^2 = {}^{*}\overline{\text{BC}}^2$, or $\overline{\text{RL}}^2 = \overline{\text{BC}}^2 - \overline{a\text{R}}^2$, *it is evident that the Surface deſcribed by* RL, *in the revolution of the figure* CAFB *about the diameter* CA, *will be equal to the Difference of the Surfaces deſcribed by the lines* Rb, R*a*. *And becauſe this happens every where; the* Solid *generated by the Elliptical Segment* CRLB, *will be equal to the Difference of the* Cylinder *generated by the parallelogram* CRbB, *and the* Cone *generated by the triangle* CR*a*.

And the Solid *generated by the Elliptical Quadrant* ABC, *will be equal to the Difference of the* Cylinder *generated by the parallelogram* CAFB, *and the* Cone *generated by the triangle* CAF.

The ſame thing will happen, if CAB *was a Circular Quadrant.*

PRO-

PROBLEM II.

FIG. 63, 64.

90. *TO deſcribe a Circumference of a Circle thro' two given points* C, G, *ſo that the arch cut off by a right line* EF *given in poſition, ſhall contain a given Angle.*

FIG. 65. 63, 64.

Let *o* be the center of the Segment which contains the given angle. Draw the ſubtenſe *ef*, and the radii *oe*, *of*; biſect in L the line joining the given points, by the perpendicular OL which meets the line EF in H, and join H, G. Then if from the interſection L, be drawn the line LB, making with EF the angle LBH equal to the angle *oef*, and the line LB be carried from the point L, ſo as to meet HG, produced if neceſſary, ſome where in D; the line drawn from the point G parallel to DL, will interſect HL in the center of the circle required.

For ſuppoſe O to be the Center; draw the radii, OG, OE, and the line LD parallel to OG. Then the triangles HLB, HOE, will be ſimilar, ſince both the angles LBH, OEH, are equal to the angle *oef*: therefore HL : HO :: LB : OE :: LD : OG. Conſequently, ſince OE = OG, LB will be = LD. *W. W. D.*

N. B. I. If the line LB can be carried from the point L, to two different points of the line HG, the Problem will have two Solutions; but when it cannot be carried to any one, the Problem is impoſſible.

II. If the given angle be a right one, the center O will coincide with the point H. For EF will then be a diameter of the circle, as well as the line LH produced; therefore their interſection H will be the center.

III. If the line CG interſects the line EF at right angles, the line LO will be parallel to EF, and LO = BE, LB = OE. Therefore the two circular Arches, deſcribed from the centers C, G, with the radius BL, will interſect each other in the center O.

FIG. 63.

IV. If the point L coincides with the point A, and alſo the line GH with GA. Then if from any point of the line HO, except the point of interſection L, the line LB be drawn as before, and carried ſomewhere upon the line CG; the reſt of the Conſtruction, as well as the Demonſtration, will be the ſame.

* Fig. 64. And if the points * C, G, coincide, ſo that the line CG becomes a tangent, and L its point of contact; then will HG become $= HL$, $OL = OG$, $HD = HL + (LD =$*$) LB$. * Art. 90. Therefore the proportion $HD : HG :: LD : OG$, will become $HL + LB : HL :: LB : LO$, in this caſe.

Fig. 65. V. The angle oef is equal to the difference of the given angle ecf to a right one. For half the arch enf will be the meaſure of the angle ecf, and half the arch rf, terminated by eo produced, will be the meaſure of the angle oef. Therefore half the ſemi-circumference e n r will be the meaſure of their difference.

Corollary I.

Fig. 63. 91. Hence, two Points E, F may be found in a right line given in poſition, from which the lines drawn to any two other given points R, C, without that line, ſhall make a right angle ERF on one ſide, and a given angle ECF on the other. For draw the line RA at right angles to EF, and take in the line paſſing thro' the points C, A, produced if neceſſary, the point G, ſuch that ∺ $CA : AR : AG$; the circumference of a circle, deſcribed thro' the points G, C, ſuch that the arch ECF may contain the given angle, will cut the line EF in the Points requir'd. For by the property of the circle, $EAF = CAG$ equal to $\overline{AR}^2$ by conſtruction. Therefore ∺ $EA : AR : AF$; conſequently the angle ERF is a right one.

Corollary II.

Fig. 66, 67. 92. If any two conjugate Diameters Aa, Bb, of an Ellipſis or Hyperbola, are given; there may be found two others, which ſhall contain a given Angle. For if in one of thoſe given diameters as Aa, produced in the Ellipſis, you take the part AG equal to half its parameter, that is $= \frac{\overline{BC}^2}{AC}$, and thro' the vertex A, you draw the line EF parallel to bB: then if thro' the points C, G, be deſcribed a circumference of a Circle, ſuch that the arch ECF may contain the given angle; the lines Nn, Mm, drawn thro' the center C of the Figure, and thro' the interſections E, F, will of the circumference and the line EF will be the Diameters required. For becauſe of the Circle, CAG = EAF equal to $\overline{BC}^2$ by conſtruction. Therefore the lines Nn, Mm, will be * two conjugate Diameters, and NCM will be equal to the given angle. * Art. 89.

If the ſemi-ordinate AP be drawn to the diameter Mm, and CM, Cm, are made equal to the mean proportional between CP, CF; the points M, m, will be * in the Curve. In the * Art. 54.
ſame manner will be found the interſections N, n, of the Diameter Nn, and the Curve.

Corollary III.

93. THE two Axes may alſo be found, in the following Fig. 68, 69.
manner. Thro' the vertex A of one of the given diameters, draw the line Ak at right angles to its conjugate Bb, and upon the middle of CG (the difference in the Ellipſis, or the ſum in the Hyperbola, of AC and $\frac{\overline{BC}^2}{AC}$) erect a perpendicular, and from its interſection O with the line Ak as a center, deſcribe a circumference of a Circle thro' the center C of the Section; Then the lines Mm, Nn, drawn thro' the center C and the interſections K, k, of the circumference with the line kK, will be the Axes required. For by the property of the Circle, GAC = kAK; and becauſe of the right angled ſimilar triangles KAF, kAE, we have AK : AF :: AE : Ak, or kAK = EAF equal to $\overline{BC}^2$ by conſtruction. Therefore the lines Nn, Mm, will be two * conjugate Diameters, and KCk will be a right angle. * Art. 89.

Corollary IV.

94. IN the Parabola, if in the diameter AP, produced towards Fig. 70.
its vertex A, be taken the line AO, equal to half its parameter, and from the point O as a center, be deſcribed a circular arch EF, ſuch that each of the angles OEF, OFE (ſuppoſing the line EF to be parallel to the ordinates of the diameter AP) be equal to the given angle, contained by the diameter ſought and its ordinates. Then the lines drawn thro' the interſection E or F,

of the arch with the line EF, parallel to the diameter AP, will be the Diameter required.

For if from the vertex's A, M, be drawn the ſemi-ordinates AQ, * *Art.* 54. MP, to the diameters MQ, AP; then will MQ = *MF = AP * *Art.* 39. $= x$, AF = PM $= y$; and becauſe $2px = {}^*yy$, we ſhall have FQ $(2x)$: AF (y) :: AF (y) :: AO (p). Therefore ſince the adjacent ſides of the equal angles QFA, OAF, are proportional, the triangles QFA, FAO, will be ſimilar. Conſequently the angle FQA will be equal to the given angle DFO.

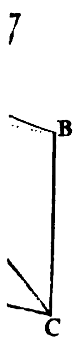
If the circular arch EF only touches the line EF in D, the diameters EL, FM, will coincide with the line Dp, which will be the Axis. For drawing Ap perpendicular to Dp, the triangles ADO, ApD, will be ſimilar; therefore pD : AD :: AD : AO, or $2px = yy$.

THEOREM XV.

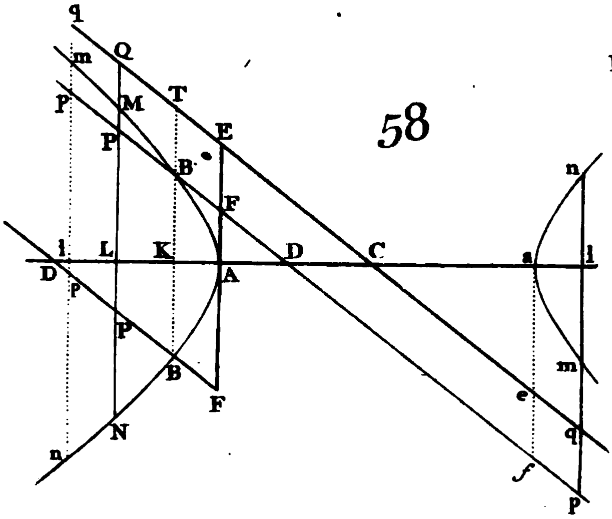
F. 71, 72, 73. 95. I*F from a point* E, *taken in the Curve of any Conic-Section, lines are drawn upon the ſides of a trapezium* ABCD *inſcribed in the Section, parallel to right lines given in poſition; the Rectangle* SET *of thoſe falling on the oppoſite ſides, will be to the Rectangle* QER *of thoſe falling on the other oppoſite ſides, in a given ratio.*

Fig. 71. *Caſe* 1. Let the ſides AB, DC, and the lines ER, EQ, be parallel betwixt themſelves, and the lines ES, ET, parallel to the ſide BC. If EQ be produced till it meets the Curve in K; then will KQ = ER; becauſe the diameter LV, biſecting the parallels AB, DC, will likewiſe biſect EK and RQ. Therefore the Rectangle EQK or REQ will be to the Rectangle CQB or TES, in the given ratio; of the Squares * of the tangents parallel to thoſe lines. * *Art.* 50.

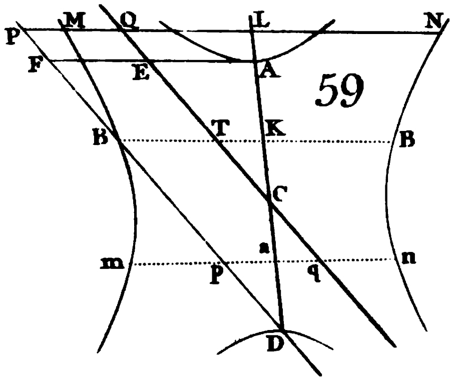
Fig. 72. *Caſe* 2. Suppoſe that the lines AB, ER, EQ, are only parallel, and the lines ES, ET, BC. If Ct, Dn, are drawn parallel to AB, interſecting the Curve in d, n, and the lines ET, CB, in t, N: Then if the line joining the points A, d, interſects EQ, DN, in r, M; the ſimilar triangles CTt, DCN, will give Ct or EQ : Tt :: DN : CN, and Rr : DM :: Ar : AM, as BQ = SE to BN (becauſe of the parallel lines AB, RQ, DN) or Rr : SE :: DM : BN, and multiplying orderly EQ x Rr : Tt x SE ::



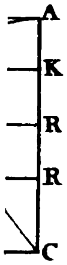
7

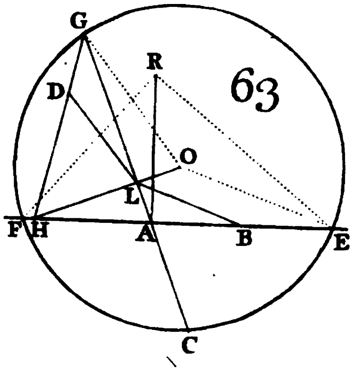


58

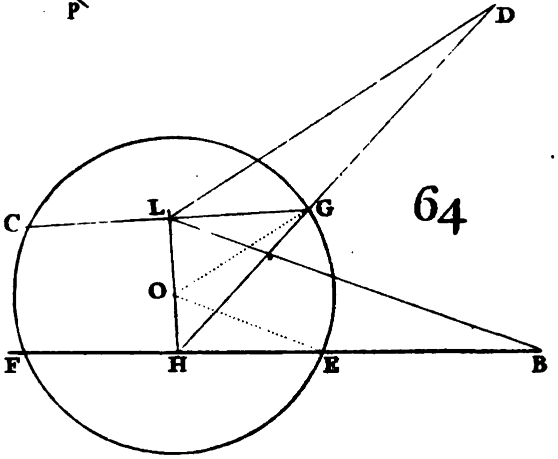


59

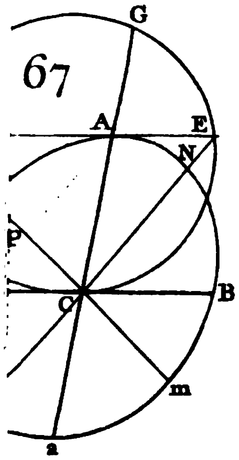




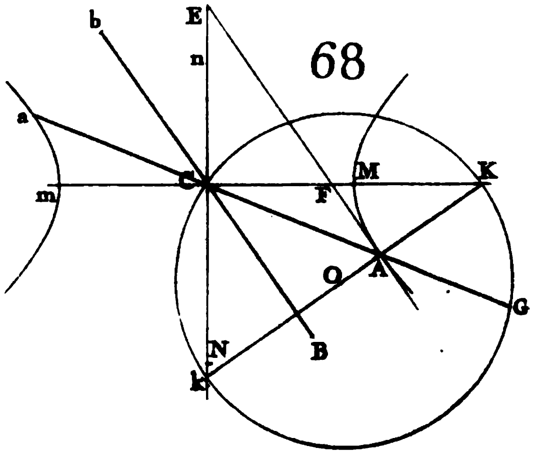
63



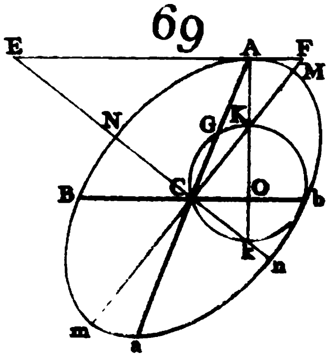
64



67



68



69

UNIV. OF

:: NDM : CNB, as rEQ to SEt, by the firſt Caſe; and by diviſion $EQ\times\overline{rE-Rr}:SE\times\overline{Et-Tt}$, or REQ : SET :: NDM : CNB. But Nn = DM, and NDM = DNn; therefore REQ : SET :: DNn : CNB.

Caſe 3. Suppoſe laſtly, that the four lines ER, EQ, ES, ET, Fig. 73. make given angles with the ſides of the trapezium, or, which amounts to the ſame, are parallel to lines given in poſition. If Eq be drawn parallel to AB, meeting AD in r, and st parallel to BC meeting BA, CD, in s, t: Then the ratio of EQ to Eq, of ER to Er, of ES to Es, and of ET to Et, will be given; becauſe all the angles of the triangles qEQ, REr, sES, tET, are given; therefore the compounded ratio's of QER to qEr, and SET to sEt, are given. Conſequently the ratio of QER to SET.

COROLLARY I.

96. HEnce may be found ſuch a Point E (the reſt as before) Fig. 74. that the Rectangles QER, SET, are in a given ratio. For drawing a line CH, from any angular point as C, meeting the line BA produced in H, and AD in I: Then becauſe all the angles of the figure, are given; the ratio's of EC to EQ and EC to ET, and therefore the ratio of EQ to ET, will be likewiſe given. Alſo taking the ratio of EQ to ET, from the given ratio of QER to SET; the ratio of ER to ES, will be given; and adding the given ratio's of EI to ER and ES to EH; the ratio of EI to EH and conſequently the point E will be given.

COROLLARY II.

97. IF a Parallelogram BQES, whoſe oppoſite angular points Fig. 72. B, E, touch the Curve of any Conic-Section, be given, and from the interſections C, A, of the adjacent ſides of one of thoſe points, lines are drawn to any point D of the Curve: The parts ER, ET, of the other ſides, cut off by thoſe lines CD, AD, will be always to each other in a given ratio. For dividing the antecedents of the proportion * REQ : SET :: NDM : CNB * *Art.* 95. *Caſe* 2. by EQ, and the conſequents by ES; then will ER : ET :: NDM

$\frac{NDM}{EQ} : \frac{CNB}{ES}$. But the ratio of NDM to CNB, as well as the lines EQ, ES are given. Conſequently the ratio of ER to ET, is given alſo.

COROLLARY III.

98. HEnce if from the points A, C, lines are drawn to any other point as d of the Curve, meeting the Sides EQ ES of the parallelogram, in r and t; then will ER : ET :: Er : Et: $\frac{NDM}{EQ} : \frac{CNB}{ES}$. Likewiſe the ratio of the lines ER, ET, making
Fig. 73. given angles with the Sides Eq, Es, of the parallelogram BsEq will be given; ſince the ratio of Er to ER, of Et to ET, and of
* Art. 39. Er to * Et, are given.

COROLLARY IV.

Fig. 72. 99. IF the point d coincides with the point C, the line Ad will become the line AC, and the line Cd will only touch the Curve in C; if therefore AC interſects the line EQ in m, we ſhall have ER : ET :: Em : Et. But if the line DC coincides with the line AB; the lines AD, BC, will then become two tangents in A and B, and the line ET will become = ES. Therefore the ratio of $\overline{ES}^2$ to REQ will then be given.

COROLLARY V.

Fig. 75. 100. HEnce (the reſt as before) may be found the Locus of all the points E. For drawing a tangent, * thro' one of the
* Art. 99. angular points B of the given trapezium ABCD, and from another point A, the line AH parallel to that tangent; and finding its inter-
* Art. 96. ſection * H with the Curve; then the line BF drawn thro' the middle G of AH, will be a diameter, which will be determined by finding its vertex F. Therefore the rectangle BGF will be to $\overline{GH}^2$ as BF is to its parameter. Conſequently the Curve may be deſcribed.

THEOREM XVI.

101. I*N any Conic-Section, the Segments* BMEO, DMFO, *terminated by the Curve, and by two lines joining the extremities of two parallel lines* BD,EF, *terminated by the Section, will be equal.* FIG. 76, 77.

For becauſe the diameter HK biſecting the lines EF, BD, will likewiſe biſect all the lines MM, MM parallel to them, as well as the ſame lines, produced if neceſſary, and terminated by the lines FD, EB. Therefore, ſince the correſpondent lines MO are always equal to each other; the Segments BMEO, DMFO will be equal.

COROLLARY I.

102. HEnce, adding to, or ſubſtracting from, the equal triangles DEB, BFD, the equal ſegments EMBO, FMDO; the Spaces DBME, BDMF, will be equal.

COROLLARY II.

103. IF the ſegment BAD be added to the equal ſpaces EMBD, FMDB; the Segment EBAD will be equal to the Segment FDAB. But if the line BD, inſtead of cutting only touches the Section, in A; the Segment EBAE will be equal to the Segment FDAF. FIG. 76.

COROLLARY III.

104. HEnce, in any given Conic-Section, may be found two Segments EBAE, FDAF, ſuch that each of them be equal to a given Segment BAD. For drawing BF parallel to AD, and DE parallel to AB; then the ſegments ABA, DMFOD, being between the parallels AD, BF, as well as the triangles BAD, ADF, will be * equal. Therefore FDAF= BAD. In the ſame manner will be proved that EBAE=BAD.

* *Art.* 101.

THEOREM XVII.

105. THE *Sectors* DCF, BCE *of an Ellipſis or Hyperbola,* F. 78,79,80. *terminated by lines drawn from the center* C, *to the interſections of any two parallel lines* BD, EF, *with the Curve, are equal.*

For drawing the diameter CA thro' the middles H, K, of the parallel lines BD, EF; and ſuppoſing that the line BD, produced if neceſſary, meets the lines CE, CF, in n and r. It is evident that the ſpaces EBn, FDr, being the differences between the equal trapeziums EnHK, FrHK, and the equal ſegments EBHK, FDHK, are equal. But the triangles CBn, CDr, having equal baſes Bn, Dr, and the ſame altitude CH, are likewiſe equal. Therefore the Sector DCF is equal to the Sector BCE.

COROLLARY I.

106. IF the line BD be ſuppoſed to move parallel to itſelf, till Fig. 78, 79. it touches only the Curve in A; it is evident, that the Sector ACF will be equal to the Sector ACE.

COROLLARY II.

107. HEnce if the Diameter CA, be drawn thro' the middle K of the line joining the Legs of the ſector ECF; the Sectors ECA, FCA will be equal.

COROLLARY III.

Fig. 81. 108. IF from the points F, D, B, E, where the lines terminating equal Sectors FCD, BCE, meet the Curve of an Hyperbola, lines are drawn parallel to one of the aſymptotes CT, and meeting the other in G, H, K, L; then will CG : CH :: CK : CL. For the lines drawn thro' the points D, B, and F, E,
* *Art.* 101. meeting the aſymptotes in Q, M, and P, N, will be parallel, *
and drawing DS, FT, parallel to the aſymptote CL, meeting the other CT in S, T; the triangles DQS, MBK and FPT, NEL
* *Art.* 82. will be ſimilar and equal*. Therefore TF : SD :: TP :
* *Art.* 80. QS. But * becauſe LE : BK :: CK : CL, and TF = CG, SD = CH,

=CH, TP=LE, QS=MK; it follows that CG: CH :: CK : CL.

COROLLARY IV.

109. HEnce, if the parts CG, CH, CK are in a continual geometrical Proportion, and the lines drawn thro' the points G, H, K, meet the Curve in F, D, B; the Sectors FCD, and DCB, will be equal. For if in the preceding Proportion, CH be ſuppoſed=CK, and CL to become=CK; then will CG: CH :: CH : CK, and the Sector DCE will become the Sector BCD=DCF.

COROLLARY V.

110. IF in the Aſymptote CN of an Hyperbola, you take as many parts CG, CH, CK, CL, *&c.* as you pleaſe, in a continual geometrical progreſſion, and from the points G, H, K, L, *&c.* you draw lines parallel to the other aſymptote CT, meeting the Curve in F, D, B, E, *&c.* the Sectors FCD, DCB, BCE, *&c.* will be equal. For becauſe ∺ CG : CH : CK, the Sectors FCD, DCB, will be equal: and ſince ∺ CH : CK : CL, the Sectors DCB, BCE, will be equal, and ſo *ad infinitum.*

COROLLARY VI.

111. THerefore, if CH be the firſt of two mean proportionals between CG and CL; the Sector FCD will be to the Sector FCE, as 1 to 3. And in general, if *m* expreſſes any whole number, and CH be the firſt of ſo many mean proportionals between CG and CL, as the number $m-1$ contains units: The Sector FCD will be to the Sector FCE, as 1 to *m.*

COROLLARY VII.

112. IF CG : CH :: $\overline{CK}^2$: $\overline{CL}^2$; the Sector FCD will be to the Sector BCE, as 2 to 1. For if CV be taken ſuch that ∺ CV : CK : CL, or CV×CL=$\overline{CK}^2$, and the line VO be drawn parallel to the aſymptote CT, meeting the Hyperbolical Curve in O; the Sector FCD will be equal to the Sector OCE, * becauſe * Art. 108.

$CG : CH :: (\overline{CK}^2)\, CV \times CL : \overline{CL}^2 :: CV : CL$. But the Sector
* Art. 109. BCE is equal to the Sector * BCO, ſince $CV \times CL = \overline{CK}^2$. Therefore $FCD : BCE :: 2 : 1$.

If $CG : CH :: \overline{CK}^3 : \overline{CL}^3$; the Sector FCD will be to the Sector BCE, as 3 to 1. For if $CV \times \overline{CL}^3 = \overline{CK}^3$, the Sector OCB will be to the Sector BCE, as 2 to 1, ſince $CV : CK :: \overline{CK}^2 : \overline{CL}^2$, or by compoſition $OCB + BCE$, or $OCE : BCE :: 3 : 1$. But becauſe $CG : CH :: \overline{CK}^3\, (= CV \times \overline{CL}^2) : \overline{CL}^3 :: CV : CL$; the Sector FCD will be equal to the Sector OCE. Therefore OCE or $FCD : BCE :: 3 : 1$.

In general if m expreſſes any whole number, and $CG : CH :: \overline{CK}^m : \overline{CL}^m$; the Sector FCD will be to the Sector BCE, as m to 1.

COROLLARY VIII.

113. IF there be taken $\overline{CK}^m : \overline{CL}^m :: \overline{CS}^n : \overline{CT}^n$; the Sector BCE will be to the Sector DCF, as n is to m. For if
* Art. 112. $CV : CK :: \overline{CK}^m : \overline{CL}^m$; then will * $BCE : OCB :: 1 : m$, and $OCB : DCF :: n : 1$; *ex æquo* $BCE : DCF :: n : m$.

If $\sqrt[m]{\overline{CK}} : \sqrt[m]{\overline{CL}} :: \sqrt[n]{\overline{CS}} : \sqrt[n]{\overline{CT}}$; it may be proved in the ſame manner as before, that $BCE : DCF :: m : n$.

COROLLARY IX.

114. IT is evident, that each Hyperbolical Trapezium DHKB is equal to its correſpondent Sector DCB. For taking
* Art. 86. the common part ACH from the equal triangles *BCK, DCH, and adding to them the curvilinear Space ABD; then will $DHKB = DCB$.

COROLLARY X.

115. HEnce it is evident, that whatever we have demonſtrated of the Sectors of an Hyperbola, ought likewiſe to be underſtood of the Hyperbolical Trapezium's DHKB, &c.

THEOREM

THEOREM XVIII.

116. L*ET there be two Hyperbolical Curves* AM, AN, or BM, DN, *which have the ſame center* C, *and the ſame line* AC *for the ſemi-conjugate diameter to* BC, DC. *If the lines* CN, CM *are drawn from the center to the extremities of any ſemi-ordinate* PN; *the Sectors* CAM, CAN, *or* CBM, CDN, *will be to each other* as BC to DC. Fig. 82, 83

For $\overline{PM}^2 : \overline{BC}^2 :: \overline{CP}^2 \mp \overline{AC}^2 : \overline{AC}^2 :: \overline{PN}^2 : \overline{DC}^2$, or PM : PN :: BC : DC; and becauſe this happens every where, and the triangles CPM, CPN having the ſame baſe, are as their altitudes PM, PN; it follows that CAM : CAN or CBM : CDN :: BC : DC. *W. W. D.*

DEFINITION 16.

TWO Curvilinear Figures are ſaid to be ſimilar, when ſimilar rectilinear Figures can always be ſimilarly inſcribed in each of them.

COROLLARY.

117. HEnce it follows, that ſimilar Curvilinear Figures are as the Squares of their homologous Sides. For there may be conceived an infinite number of Poligons inſcribed in thoſe Figures, ſo that the laſt coincides with the Figure itſelf.

THEOREM XIX.

118. I*F the Semi-diameters* $CB = a$, $CD = b$, *of two Hyperbola's* BL, DN, *are proportional to their parameters* p, q, *that is, if* $\frac{a}{p} = \frac{b}{q}$: *and from the interſections* L, N (*of a line drawn from the center with the Curves*) *the ſemi-ordinates* Lp, NP *are drawn to the ſemi-conjugate* CA; *the Segments* CBLp, CDNP *will be to each other as* $\overline{CB}^2$ *to* $\overline{CD}^2$. Fig. 83.

* Art. 39. For ſince * $\frac{a}{p}\overline{Cp}^2=\overline{pL}^2-\overline{CB}^2$, and $\frac{b}{q}\overline{CP}^2=\overline{PN}^2-\overline{CD}^2$; we ſhall have $\frac{a}{p}\overline{Cp}^2 : \frac{b}{q}\overline{CP}^2$ or $\overline{Cp}^2 : \overline{CP}^2 :: \overline{pL}^2-\overline{CB}^2 : \overline{PN}^2-\overline{CD}^2$, as $\overline{pL}^2$ to $\overline{PN}^2$, becauſe of the ſimilar triangles CpL, CPN; and by diviſion, $\overline{CB}^2 : \overline{CD}^2 :: \overline{pL}^2 : \overline{PN}^2 :: \overline{CL}^2 : \overline{CN}^2$. Therefore the lines BL, DN are always parallel. Alſo the Sectors CBL, CDN are ſimilar, conſequently they are as the Squares $\overline{CB}^2$, $\overline{CD}^2$, as well as the ſimilar triangles CpL, CPN. Wherefore CBLp : CDNP :: $\overline{CB}^2 : \overline{CD}^2$. *W. W. D.*

COROLLARY I.

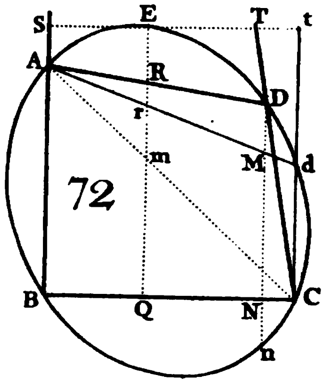
Fig. 84. 119. IF from the common center C of the ſimilar Hyperbola's AL, BN, the line CN be drawn at pleaſure, and from the points A, B, L, N, where the ſemi-diameter CB and the line CN meet the Curves, the lines AD, BE, LF, NG, are drawn upon the common aſymptote CG, and parallel to the other CS; the Hyperbolical Trapezium's DALF, EBNG, will be to each other * as $\overline{CA}^2$ to $\overline{CB}^2$, or as $\overline{CD}^2$ to $\overline{CE}^2$.

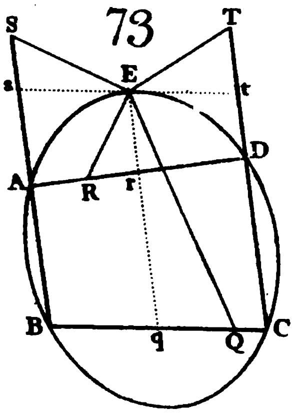
* Art. 114. 118.

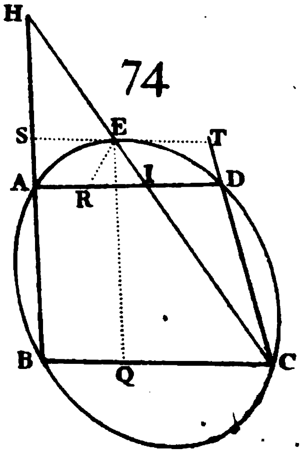
COROLLARY II.

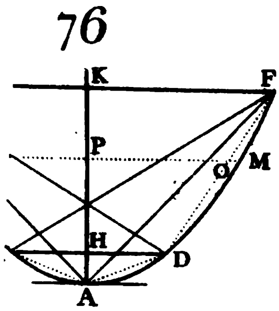
120. IF the line FL produced, meets the Curve BN in M, and EB the Curve AL in K; we ſhall have EKLF : EBMF

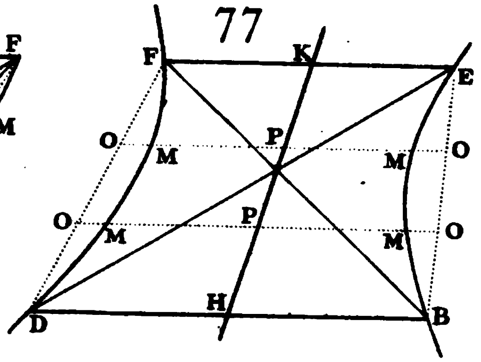
* Art. 80. :: $\overline{CD}^2 : \overline{CE}^2$. For FL* $=\frac{CD\times DA}{CF}$, and FM $=\frac{CE\times EB}{CF}$; therefore FL : FM :: $\frac{CD\times DA}{CF} : \frac{CE\times EB}{CF}$, as $\overline{CD}^2$ to $\overline{CE}^2$, becauſe of the ſimilar triangles CDA, CEB; and ſince this happens every where, it follows that EKLF : EBMF :: $\overline{CD}^2 : \overline{CE}^2$.

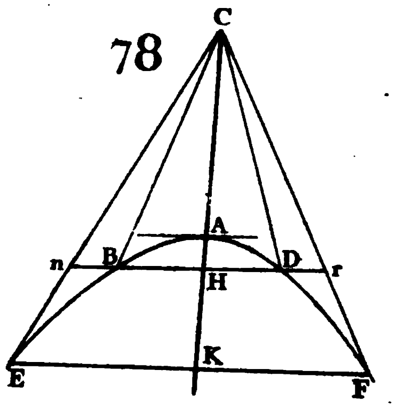

72

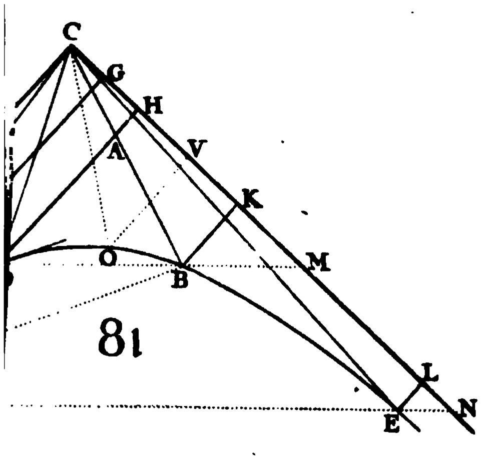

73

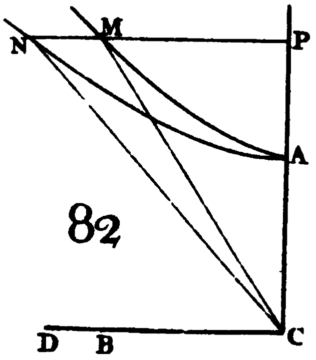

74

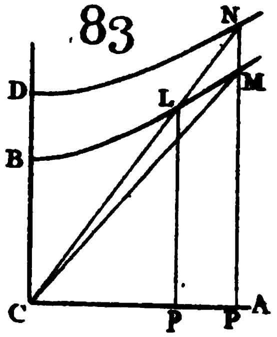

76


77


78


81


82


83

UNIV. OF MICH.

COROLLARY III.

121. IF $\overline{CD}^2 : \overline{CE}^2 :: m : n$, there may be found an Hyperbolical Trapezium RSVT in the Hyperbola VTL, equal to a given Trapezium EBMF in the Hyperbola BN, the point R in the aſymptote CS being given.

For, becauſe EKLF : EBMF :: $m : n$, if CS be taken ſuch that $\sqrt[m]{CE} : \sqrt[m]{CF} :: \sqrt[n]{CR} : \sqrt[n]{CS}$, or $\sqrt[m]{\overline{CE}^n} : \sqrt[m]{\overline{CF}^n} :: CR : CS$; the lines RT, SV, drawn parallel to the aſymptote CG, will determine * the Trapezium required. * *Art.* 113.

THEOREM XX.

122. I*F three tangents to any Conic-Section, interſect each other in* L, T, H, *and from one of thoſe interſections as* T, *be drawn a diameter* TA, *and from the vertex* A *the tangent* EF *meeting* HT, LT, *in* F, E; *then if in the Ellipſis and Hyperbola the conjugate diameter to* TA, *meets likewiſe thoſe tangents in* K, k; *we ſhall have* KH : KF :: kE : kL. FIG. 85, 86.

For if from the points H, L, the lines HB, LD, are drawn parallel to EF, and meet the diameter in B, D: then will CB : CA :: KH : KF, and CA : CD :: kE : kL. But CB : CA :: * CA : CD; therefore KH : KF :: kE : kL. * *Art.* 61.

And in the Parabola, if thro' the points of contact N, M, m, *of the tangents* NT, ML, HF, *be drawn the ſemi-ordinates* NQ, MP, mp; *then will* 2BH=QN+pm, *and* 2DL=QN−PM. FIG. 87.

For let PM$=y$, AP*=AE$=x$, QN$=z$, AQ=*AT$=u$, DL$=a$; and the parameter of the diameter AP, $=1$ then will PM : PE :: LD : DE $= \frac{2ax}{y}$, or DE+EA=, DA$= \frac{2ax}{y} + x$, and QN : QT :: DL : DT$= \frac{2au}{z}$, or TA−TD=DA$=u - \frac{2au}{z} = \frac{2ax}{y} + x$, or becauſe $x = yy$, $u = zz$; $zz - 2az = 2ay + yy$. From whence we get $zz - yy = 2ay + 2az$, or by dividing by $y + z$, $z - y = 2a = 2$DL. * *Art.* 54.

In like manner may be proved, that 2BH=QN+pm.

COR.

COROLLARY I.

Fig. 85, 86. * *Art.* 122.

123. IN the Ellipſis and Hyperbola, if there be drawn another tangent, meeting the tangents TH, TE, in l, h; we ſhall have $KH:Kl::kh:kL$. For $KF:Kl::$ * $kh:kE$, and $KH:KF::kE:kL$; *ex æquo* $KH:Kl::kh:kL$.

COROLLARY II.

Fig. 88, 89. Fig. 88. * *Art.* 122. * *Art.* 123.

124. HEnce the diameter DB biſecting in *u* the diagonal Ll of a trapezium LHlh, whoſe ſides touch any Conic-Section, will likewiſe biſect the other diagonal Hh of that trapezium in r. For if in the Ellipſis and Hyperbola the lines LD, ld, hb, HB are drawn parallel to the diameter * Kk, and if $CK=Ck=a$, $DL=dl=b$, $BH=x$, $bh=y$; the parallel lines KC, HB, ld, and kC, hb, LD, will give $KH:Kl::$ $(CK-BH:CK-dl)\ a-x:a-b$; aud $kh:kL::(Ck-bh:Ck-DL)\ a-y:a-b$. But * $KH:Kl::kh:kL$; therefore $a-x:a-b::a-y:a-b$, or $a-x=a-y$. Hence $x=y$ $(BH=bh)$: Conſequently $hr=rH$.

Fig. 89. * *Art.* 122.

And in the Parabola, if from the points of contact, the ſemi-ordinates NQ, MP, nq, mp, are drawn to the diameter DQ: then becauſe $2DL=$ * $qn-PM$, $2dl=QN-pm$, $2BH=QN+PM$; $2bh=qn+pm$; if $(DL=dl)$ $qn-PM=QN-pm$, or $qn+pm=QN+PM$; we ſhall have $BH=bh$: Conſequently $hr=rH$.

THEOREM XXI.

Fig. 90.

125. IF *two given Angles* BAK, BDK, *revolve about their angular points* A, D, *fixed upon a Plane: I ſay the interſection* B *of the legs* AB, DB *of thoſe Angles, will deſcribe a Conic-Section, whilſt the interſection* K, *of the other legs* AK, DK, *deſcribes a right line* KF *given in poſition, and which does not paſs thro' either of the points* A, D.

Let the legs AK, DK paſs thro' a given point F of the line KF, and let C be the interſection of the other legs; draw the lines CR, CS, ſo that the angles ACR, DCS, be equal to the angles AFK, DFK. Then becauſe of the equal angles KAB, FAC, and KDB, FDC, the angle KAF will be equal to the angle CAR, and KDF=CDB, and ſo the triangles AFK, ACR, and DFK, DCS, will be ſimilar; wherefore CR : FK :: AC : AF, and FK : CS :: DF : DC; and multiplying orderly, CR : CS :: AC×DF : AF×DC. But ſince the lines AC, DF, AF, DC, are given, the ratio of CR to CS will be given alſo; and as the angles DCS, ACR, are likewiſe given, the point B will always be in the Curve of a *Conic-Section paſſing thro' the points A, C, D. *W. W. D.* * *Art.* 98.

N. B. When the line FK paſſes thro' one of the fixed points as D, it will coincide with the leg DK, alſo the other leg DB will be given in poſition, and therefore the interſection B will always be in the right line DB.

If the legs of one of the moveable angles coincide with each other, the Propoſition will remain the ſame.

COROLLARY I.

126. HEnce, if the leg AB coincides with the line AD, the leg DB will become a tangent at D to the Section; and if the leg DB coincides with the line DA, the leg AB will become a tangent at A; ſince in both Caſes, thoſe legs meet the Curve but in one point, and fall intirely without the ſame.

COROLLARY II.

127. IF the legs AK, DK, coincide with the line DA, ſo that the moveable Angles BAK, BDK, become the angles VAL, EDL; the Angle E made by the interſection of the other legs, produced at the other ſide of the points A, D, will be the difference of the moveable Angles to two right ones.

COROLLARY III.

128. IF thro' the points A, D, there be deſcribed a Circumference of a Circle, ſo that the arch AGD, towards the line EK, may contain the difference of the two moveable angles to four right ones; and if the legs AL, DF of the moveable Angles, paſs thro' the interſection G of the circumference, and the diameter HG perpendicular to FK; the other legs AX, DN, will be parallel to one of the axes. For becauſe the difference of the Fig. 91.

P move-

moveable angles XAG, GDN, to the angles GAD, GDA, is equal to two right ones by Hyp:, the legs AX, DN will be parallel. And if the interſections N, L, X, F, of the legs, the Curve, and the line FK, as well as the point H, are joined to the points A, D, the right angled ſimilar triangles HAG, GTL, and HDG, GTF, will give $AG \times GL = HG \times GT = DG \times GF$. But becauſe the adjacent ſides of the oppoſite angles at G are proportional, the triangles AGF, DGL will be ſimilar; and ſo the angles GAF, GDL, or their equals NAX, NDX, as well as their alternate ones AND, DXA, will be equal. Therefore the perpendicular QM, which biſects one of the lines DN, will alſo biſect the other AX, and conſequently will be * one of the axes.

* *Art.* 73.

THEOREM XXII.

FIG. 92.

129. EVERY *Thing elſe being as before: I ſay that in the Ellipſis or Hyperbola, the parts* HT, TG *of the diameter* HG, *produced if neceſſary, terminated by the line* FT, *expreſs the Ratio of the Axis, parallel to the legs* DN, AX, *to its Parameter.*

Let the leg AB of the moveable angle KAB, be perpendicular to the lines AX, DN; from the point F draw FV, FS, and thro' the interſection K the line KR at right angles to KA, AL and DF produced. Then if $TH=a$, $TG=p$, $TF=r$, $TL=s$, $LF=c$, $GF=n$, $GL=m$, $FK=v$, $AP=d$, $DP=x$, $PB=y$; the ſimilar triangles GTF, KRF, GDH, will give $\frac{rv}{n} = FR$, $\frac{pv}{n} = KR$, and $GD = \frac{ap-pp}{n}$. Alſo $DG+GF+FR = \frac{ap+rr+rv}{n}$, ſince $nn=rr+pp$; and becauſe of the ſimilar triangles DRK, DPB, we have ($DR:RK::DP:PB$) $apy-rry+rvy=pvx$, or $v = \frac{apy+rry}{px-ry}$. But ſince the angles XAL, BAK are equal, and XAB is a right angle by Hyp:; the angle KAG will be a right angle alſo, and ſo the lines KA, AH will coincide. Therefore the ſimilar triangles HTK, FVK, GTL, LSF, will give 1.° $GL:LT::KF:FV=\frac{ap-rs}{m}$, 2.° $GL:GT::LF:FS=VA=\frac{pc}{m}$, 3.° $LT:TG::HT:TK=\frac{ap}{s}$. Alſo $TK-TF=v=\frac{ap-rs}{s}=\frac{apy+rry}{px-ry}$; or $apx-rsx=acy$, becauſe $r+s=c$.

And the ſimilar triangles AVF, APN, will give $AV:VF::AP:PN=\frac{d}{pc}\times\overline{ap-rs}$; wherefore $DPN=\frac{dx}{pc}\times\overline{ap-rs}$, or $\frac{pc}{d}\times DPN=apx-rsx=acy$. Conſequently * $DPN:APB\,(dy)$ * Art. 47.
$::a:p$. *W. W. D.*

COROLLARY I.

130. WHEN the line KT falls without the Circle AGD, the interſection B will deſcribe the Curve of an Ellipſis. For the angle AKD being always leſs than the complement of the two moveable Angles to four right ones, the legs AB, DB will always interſect each other whereſoever the interſection K falls. Therefore the Curve deſcribed by the interſection B, will return into itſelf in this caſe, and conſequently will be an Ellipſis.

When the line KT cuts the Circle AGD, the interſection B will deſcribe the Curve of an Hyperbola; becauſe when the interſection K falls in the interſections of the line KT with the circumference; the legs AB, DB, will become * parallel between * Art. 128.
themſelves; therefore each of them will meet the Curve but in one point: and ſince there can be drawn no other two lines, which are not parallel, ſo as to meet a Conic-Section but in one point, beſides thoſe which are parallel to the aſymptotes; it follows that, &c.

And when the line KT only touches the Circle AGD, the interſection B will deſcribe the Curve of a Parabola, ſince the ratio of the parameter *p*, to the axis *a* becomes infinite in that caſe.

N. B. The Axis parallel to the lines AX, DN will be always the firſt; ſince it is always greater than its parameter in the Ellipſis,

lipſis, and is always in that angle made by the aſymptotes, in which the oppoſite Hyperbola's are.

COROLLARY II.

131. IF the Ratio of the firſt Axis to its Parameter be given,
Fig. 93. you may deſcribe the Curve of an Ellipſis or Hyperbola, thro' four given points A, B, D, E. For joining three of them as A, E, D, by right lines, and if upon A D as a chord, you deſcribe a circular arch A H D, at the other ſide of A D, with reſpect to the point E, which may contain the angle A E D: then if you make paſs the legs of the angles B A K, B D K (equal to the complements V A D, L D E of the angles D A E, A D E to two right ones) thro' the fourth point B; the line K T, drawn thro' the interſection K of the other legs, and tangent to the Circle T t deſcribed with the center C of the former Circle, and
* *Art.* 47. whoſe radius C T is to the radius C G, as * $a \pm p$ is to $a \mp p$, will ſerve for deſcribing the Curve required.

N. B. As there can be drawn two tangents from the point K to the Circle T t, ſo there may be deſcribed the Curves of two Conic-Sections of the ſame kind thro' the four given points.

In the Parabola, the line K T is drawn a tangent to the Circle A G D.

THEOREM XXIII.

132. *THE Curves of any two Conic-Sections can cut each other*
Fig. 94. *in no more than four Points.*

Let m, H, K, n, be four contiguous interſections of the Curves m H L n, and m H F K n, draw the lines m H, n K, meeting each other in T, and make T H : T m :: R H : R m, and T K : T n :: S K : S n. Then becauſe the parts of any line T E, terminated by the
* *Art.* 53. point T and the Curves, are always as the * parts terminated by the line R S and the Curves: and as the line T E meets always the Curves in two points L, F, on one Side of the line R S by Hyp:, it follows that it will likewiſe meet always the Curves in two points on the other.

PROBLEM

PROBLEM III.

133. *TO deſcribe the Curve of a Conic-Section thro' five given points* A, B, C, D, E, *provided that but two are in a right line.* FIG. 90.

Join any three of thoſe points as A, D, E, by right lines, and thro' the two others let paſs the legs of the angles BAK, CAF, equal to the complement VAD of the angle DAE, to two right ones, and the angles BDK, CDF, equal to the complement LDE of the angle ADE: then the interſection B of the legs AB, DB, of the moveable Angles KAB, KDB, revolving about the points A, D, will deſcribe the Curve required, whilſt the interſection K of the other legs AK, DK, deſcribes the * right line KF drawn thro' the points K, F. *W. W. D.* * *Art.* 125.

Otherwiſe.

From any of the given points as E, draw the lines EQ, ET, parallel to the Sides AB, BC, of the trapezium formed by the lines joining the four others A, B, C, D, meeting the four Sides in R, Q, S, T; then if from the point C the line C*f* be drawn parallel to BA, and from its interſection *f* with ST, the line *f*r parallel to TR, meeting EQ in r; the interſection d of the lines C*f* and Ar will be in the Curve. For becauſe of the parallel lines TR, *f*r, we have ET : ER :: E*f* : Er; therefore the point d will be in the Curve * of a Conic-Section, which paſſes thro' the five given points A, B, C, D, E. FIG. 95. * *Art.* 98.

And if Au be drawn parallel to BC, and thro' its interſection u with EQ, the line u*g* parallel to RT meeting ET in *g*: then the interſection e of the lines C*g* and Au, will be alſo in the Curve. Therefore the interſection O of the lines drawn thro' the middles of the parallels AB, dC, and BC, Ae, will be the * Center, which being premiſed. * *Art.* 36.

Let the lines CB, CA, produced if neceſſary, meet the diameter mM, biſecting AB, dC, in t, P. Then if OM = Om be taken a mean proportional between OP, and Ot; the points M, m will be * in the Curve. * *Art.* 54, 56.

In like manner will be found the vertex's N, n, of the diameter N n conjugate to M m. Therefore the Curve may be deſcribed by *Article* 43.

N. B. I. When $Mt > MP$, the Section will be an Ellipſis; when $Mt < MP$, an Hyperbola; and when $Mt = MP$, it will be a Parabola.

II. Becauſe the line F K is determined by the interſections of
Fig. 90. the legs AK, DK, whilſt the others AB, DB, paſs thro' the two given points B, C, and ſince the moveable angles are formed by means of the three others A, D, E; it follows that a Conic-Section can paſs thro' no more than five given points. And as the
* *Art.* 132. Curves of two Conic-Sections can cut each other but in four * points; it follows that there can be deſcribed but one Conic-Section thro' five given Points.

PROBLEM IV.

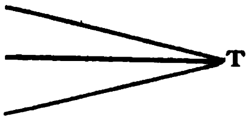
134. *TO deſcribe the Curve of a Conic-Section, thro' four given*
Fig. 96, 97. *points* A, B, C, E, *which ſhall touch a right line* bT *given in poſition.*

Caſe 1. Let C be the point of contact, as alſo one of the given points. If three of thoſe points as A, B, C, are joined by right
Fig. 96. lines, and from the fourth E you draw the lines S T, E Q parallel to BC, and AB interſecting in Q, R, S, the lines BC, AC, AB, and the tangent in T; then if in the lines S T, E Q, the points t, r, are taken ſuch that ER : ET :: Er : Et; the interſection d of the lines drawn thro' the points A, r, and C, t, will be in * the Curve. Therefore, five points A, B, C, d, E, being given,
* *Art.* 98. the Curve may be deſcribed as before.

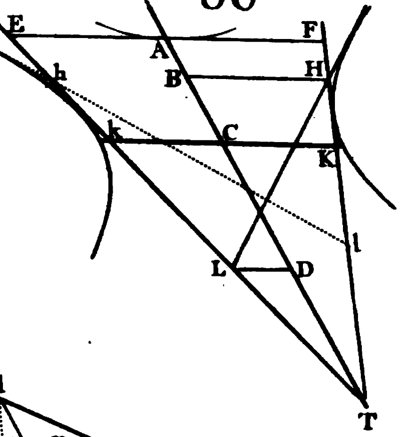
Caſe 2. Let G be the point of concourſe of the lines drawn thro'
Fig. 97. the four given points, and T, b, their interſections with the tangent. If the rectangle $AbB = aa$, $BGA = bb$, $EGC = cc$, $CTE = dd$, and $bD : DT :: a : \frac{bd}{c}$; the point D will be the point of contact. For if the line T S parallel to AB interſects the Curve in

* *Art.* 48.
* *Art.* 50.
R, S; then will *$EGC : BGA :: CTE : RTS = \frac{bbdd}{cc}$, and *

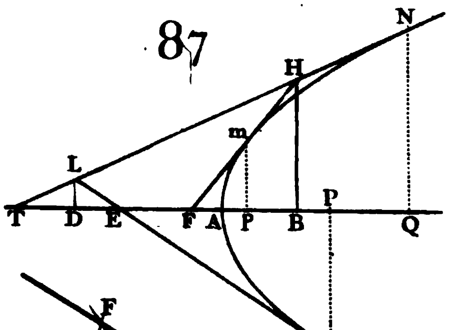
$bD : DT :: \sqrt{AbB}\ (a) : \sqrt{RTS}\ (\frac{bd}{c})$ *W. W. D.*



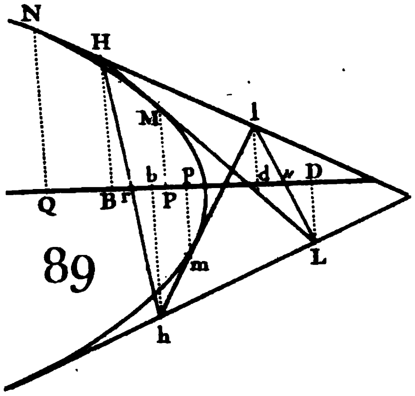
86



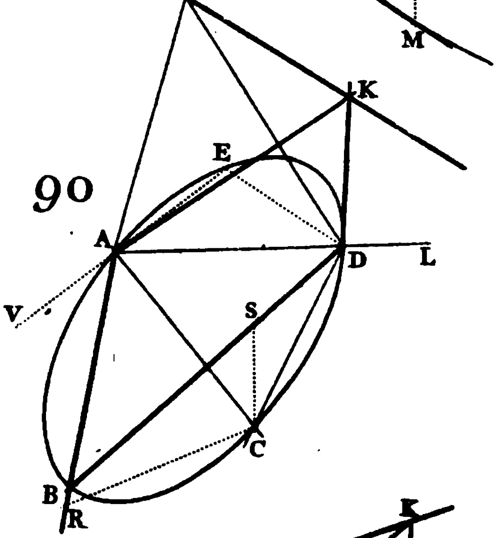
87



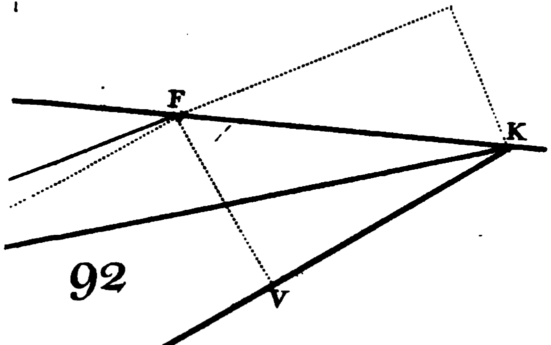
89



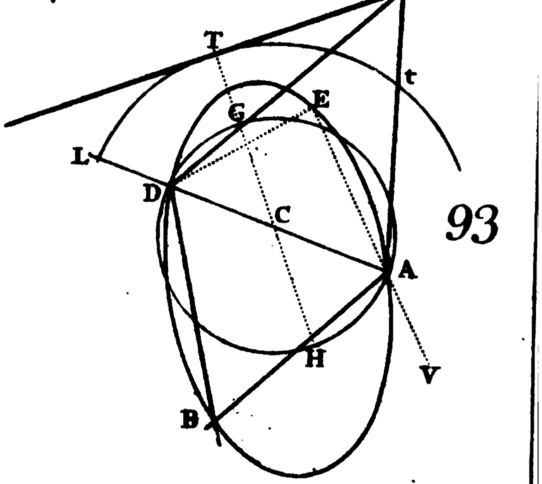
90



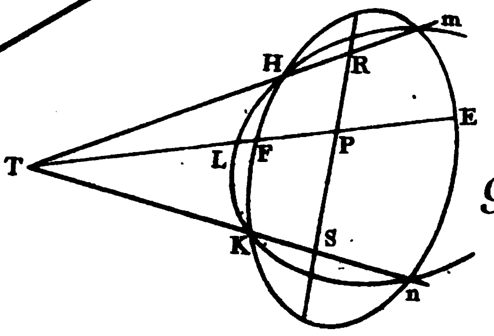
92



93



94



N. B. As the point D may be taken within or without the points T, b, except when the rectangles A bB, RTS are equal, ſo there may be deſcribed two Conic-Sections thro' four given points, and ſo as to touch a right line given in poſition. When AbB=RTS, then will bD be=DT.

PROBLEM V.

135. *To deſcribe the Curve of a Conic-Section, thro' three given points* B, C, D, *which ſhall touch two right lines* AT, FIG. 98. PT, *given in poſition.*

Draw the lines DC, DB, thro' the given points, interſecting the tangents in I, L, E, H, and make KL : KI :: $\sqrt{CLD} : \sqrt{DIC}$ and ER : RH :: $\sqrt{DEB} : \sqrt{BHD}$; then the line drawn thro' the the points R, K, * will give the points of contact A, P. Therefore the Curve may be deſcribed as before. *W. W. D.* *Art. 65.

PROBLEM VI.

136. *To deſcribe the Curve of a Conic-Section, thro' two given* FIG. 99. *points* D, E, *which ſhall touch three right lines* AT, TN, NC, *given in poſition.*

Let the line joining the given points, interſect the tangents in L, P, F. Then if KL : KF :: $\sqrt{ELD} : \sqrt{DFE}$, and F*a* : P*a* :: $\sqrt{DFE} : \sqrt{EPD}$, the points K, *a*, * will be the points of interſection of the line DE and the lines joining the points of contact B, A and A, C, and if from the point K you draw a line to the interſection N of two tangents, interſecting the third in R, and the line AC in r, and you make KR : KN :: Rr : rN; the point r * will be in the line AC. Therefore the lines drawn thro' the points r, *a*, will give the points of contact A, C, and the line drawn thro' the points K, A, will give the point of contact B: and conſequently the Curve may be deſcribed. *W. W. D.* *Art. 65. *Art. 65.

PROBLEM VII.

137. *To deſcribe the Curve of a Conic-Section, thro' a given* FIG. 100. *point* E, *which ſhall touch four right lines* NT, TM, MH, HN *given in poſition.*

The interſection *a* of the diagonals HT, NM, of the trapezium formed by thoſe tangents, will be in the ſame point as the interſection of the lines joining the points of contact A, C and B, D; therefore if the line drawn thro' that interſection *a*, and the given point E, interſects three of thoſe tangents in L, P, F, and $\overline{aF}^2 : \overline{aP}^2$ :: SFE : EPS, the point S will
* Art. 65. be in the * Curve. But if ELS : SFE :: $\overline{KL}^2 : \overline{KF}^2$, the point K will be in the line joining the points of contact B, A, from which drawing a line to the interſection N of the tangents NT, HN, meeting the tangent TM in R : and if KN : KR :: rN :
* Art. 68. rR; the point r will be * in the line joining the points of contact A, C. Therefore the line drawn thro' the points r, *a*, will give the points of contact A, C, the line drawn thro' K and A, will give the point of contact B; and laſtly the line drawn thro' the points B, *a*, will give the point of contact D. *W. W. D.*

PROBLEM VIII.

138. T*O deſcribe the Curve of a Conic-Section, which ſhall*
Fig. 101. *touch five right lines* GH, HM, TK, MF, FG, *given in poſition.*

The interſection *a* of the diagonals GN, TF of the trapezium
* Art. 67. GTNF will likewiſe be * the interſection of the lines drawn thro' the points of contact A, D and B, E; the interſection b of the diagonals GM, HF, of the trapezium GHMF will be the interſection of the lines drawn thro' the points of contact A, C and B, E; and the interſection n of the diagonals GL, HK of the trapezium GHLK, will alſo be the interſection of the lines drawn thro' the points of contact B, D and A, C. Therefore, by the help of the three given points *a*, *b*, n, the Points of contact will be found. *W. W. D.*

SCHOLIUM.

I. I*N the five preceding Problems, the center or a focus of a Conic-*
Fig. 102. *Section, is to be taken for two points, and an aſymptote for two tangents. For example, if the two aſymptotes* EN, BP *of the oppoſite Hyperbola's* AQ, DR, *and the tangent* EF *are given; it is plain*

plain, that the tangent EF *is parallel and equal* * *to the conjugate of the diameter* Aa, *which biſects that tangent, and conſequently paſſes thro' the point of contact.*

II. *If a point* D *of the Curve, and the aſymptotes, are given. By drawing* DB *at pleaſure, meeting the aſymptotes in* N, B, *and making* BR=DN; *the point* R *will be in the Curve.*

III. *If three points* Q, D, R *of the Curve, and the aſymptote* EN, *are given. By drawing two lines thro' theſe points, meeting the given aſymptote in* L, N: *and if* BR=DN, QP=DL; *the points* P, B, *will be in the other aſymptote, which therefore will be given in poſition.*

IV. *If the center* O *and the three points* A, B, C *of the Curve of an Ellipſis or Hyperbola are given. Draw the diameter* Mm *thro' the middle of the line* AB *joining two of thoſe points, and join the points* A, C, *then the reſt may be done as in Article* 133. FIG. 95.

V. *If the center* O, *the point* m *of the Curve, and the two tangents* TA, TB, *are given. It is evident, that the line* AB *joining the points of contact* A, B, *is biſected by the diameter* TO, *and is parallel to a line given in poſition. For taking the circular arch* atb *able to contain the angle* ATB, *and making the angle* dab *equal to the angle* QTB; *then if from the interſection* d, *of the line* ad *and the circumference, the line* dt *be drawn thro' the middle* q *of the line* ab; *the triangles* ATB, atb, *and* QTB, qtb, *will be ſimilar. Therefore,* bq : qt :: BQ : QT. *which being premiſed.* FIG. 99.

Draw a diameter thro' the given point m; *then the line drawn from the point, where that diameter meets* BA *produced (found as before* *) *parallel to the line given in poſition, will determine the points* A, B, *required.* * Art. 136.

VI. *Laſtly if the center* O *and the three tangents* AT, TN, NC, *are given, it is plain that the ratio's of the ſides of the triangles* ABT, BCN, ABC, *are given; ſince the lines* AB, AC, BC, *are parallel to lines given in poſition. Therefore if* $BT=x$, $BN=a-x$, $AB=\frac{bx}{a}$, $BC=\frac{ac-cx}{a}$, *and* $AB : BC :: d : n$, *or* $bnx=acd-cdx$; *we ſhall have* $x=\frac{acd}{bn+cd}$. *The value of* x *gives the point of contact* B; *and conſequently the points of contact* A, C, *are alſo given.*

LEMMA I.

FIG. 103. 139. *IF the angle* mtn, *and the ratio of the Sines* ds, dr, *of the two parts* mtd, dtn *of that angle, are given; the line* td *will be given in poſition.*

For if the line tc be drawn at right angles to the ſubtenſe mn of the given angle; the ſimilar triangles mct, msd, and nct, nrd, will give tc:tn::dr:dn. and tc:tn(=tm)::ds:dm. Therefore dr:dn::ds:dm conſequently the ratio of the lines md, dn, being given, the line td is alſo given in poſition. *W. W. D.*

LEMMA II

FIG. 104. 140. *THree points* A, B, C, *being given, to find a fourth* D, *ſuch that the Differences of the lines* AD, BD, CD, *ſhall be given.*

Suppoſe $BD=u$, $AD=c+u$, $CD=d+u$, $AB=a$, $BC=b$; take $BP=\frac{aa-cc}{2a}$, $BQ=\frac{bb-dd}{2b}$, and from the points P, Q, D, draw the lines PS, DF and QR, DE, perpendicular, and DS, DR parallel to the lines AB, BC. Then if $DS=FP=x$, $DR=EQ=y$; BF will be $=x-\frac{aa+cc}{2a}$, $BE=y-\frac{bb+dd}{2b}$: and becauſe of the right angled triangles AFD, BFD, and CED, BED, we have ($\overline{AD}^2=\overline{AB}^2+\overline{BD}^2+2AB\times BF$) $cu=ax$, and ($\overline{DC}^2=\overline{BC}^2+\overline{BD}^2+2BC\times BE$) $du=by$; and by exterminating u, we ſhall have $bcy=adx$. Therefore, becauſe the ratio of the lines DS (x), DR (y) is given, the line drawn from the interſection T of the lines PS and QR, to the point D will be alſo given in poſition.

If the line TD interſects the line CB in (a); the ſimilar triangles TQa, TRD, will give $aQ\,(n):aT\,(m)::DR\,(y=\frac{du}{b}):DT=z=\frac{dmu}{nb}$. Therefore, if the line rx be drawn ſuch that $rT:rx::dm:bn$, the line BD drawn from the point B parallel to xr, will interſect the line TD in the point D required. *W. W. D.*

PRO-

PROBLEM IX.

141. THE *focus* F, *and three points* N, L, M, *of the Curve of any Conic-Section being given, to find the Axis and the other Focus.* Fig. 105.

It is evident that the three lines MF (a), LF (b), NF (c), are the differences of the lines Mf (u), Lf (y), Nf (z), to the firſt axis Aa in the Ellipſis, and the ſame lines added to the firſt axis, are equal to the lines u, y, z, in the Hyperbola, viz. $a+u=$ Aa$=b+y$; $b+y=c+z$; $a+u=c+z$, or $a-b=y-u$, $c-b=y-z$, $c-a=u-z$, in the Ellipſis; and $u-a=y-b$, $y-b=z-c$, $u-a=z-c$, or $b-a=y-u$, $b-c=y-z$, $a-c=u-z$, in the Hyperbola. Therefore the other Focus f as well as the firſt Axis, may be found by the preceding Lemma.

Otherwiſe.

Join the given focus and the given points by right lines, and take in the lines NM, NL produced, the points P, D, ſuch that FN : FM :: NP : MP, and FN : FL :: ND : LD; the line FB perpendicular to the line drawn thro' the points D, P, will be the Axis, and DP the * Directrix. Therefore the reſt may be found as in Article 15. *Definit. 13.

For drawing the lines NK, LQ, MH perpendicular to DP; the ſimilar triangles DQL, DKN, and PHM, PKN, will give ND : LD :: NK : LQ as FN to FL by Conſtruction, and NP : MP :: NK : MH :: NF : MF. Therefore the line * DP is the Directrix of the Section, which paſſes thro' the points M, N, and the focus of which is the point F. *W. W. D.* * *Art.* 13.

COROLLARY I.

142. IF inſtead of three points, there were two points and a tangent, or two tangents and a point given: it is evident, that the other Focus and the firſt Axis may be found in the ſame manner as before. For example, ſuppoſe a tangent to be given inſtead of a point. Fig. 106.

If from the given focus F the line FD be drawn at right angles to the tangent DI, and DL be taken equal to DF; then the Fig. 106.

line

* *Art.* 17. line Lf will be equal to the * firſt axis; therefore its Difference to another line drawn from a given point of the Curve to the focus f, will be given.

But if three tangents EM, HN, DI, are given. From the given focus F draw lines FM, FD, FN, at right angles to thoſe tangents: then the center C of the Circle, whoſe circumference * *Art.* 28. paſſes thro' the three points M, D, N, will be likewiſe * the center of the Section.

COROLLARY II.

FIG. 105. 143. THE line FP joining the focus, and the interſection of the directrix with the line drawn thro' the two points N, M, of the Curve, will biſect the complement EFM of the angle made by the lines drawn from thoſe points to the focus. For drawing MT parallel to NE, we ſhall have $PM : PN ::$ * *Art.* 141. $MT : NF ::$ * $MF : FN$; therefore $MT = FM$. Conſequently the angle MFT is equal to the angle MTF, or equal to its alternate one EFT.

SECTION II.

Of CONIC-SECTIONS *conſidered in the Solid.*

DEFINITIONS.

I.

FIG. 113. IF a right line KY drawn thro' any immoveable point K, aſſumed without a Plane, in which the circumference of a circle YxXy is deſcribed, be indefinitely produced both ways, and moves quite round that circumference; then each of the ſuperficies deſcribed by the motion of the line KY, is call'd a *Conic-Superficies*; and both together, *Oppoſite Conic-Superficies.*

2.

The immoveable point K, common to both the oppoſite Superficies, is call'd the *vertex.* And the Circle Y*x*X*y* the *Baſe.*

3.

The Solid comprehended between the baſe Y*x*X*y*, and the part of the Conic-Superficies between the baſe and the vertex, is called a *Cone.*

4.

Any line as KY drawn from the vertex K to a point Y in the baſe, is called a *Side* of the Cone.

5.

The line drawn from the vertex thro' the center of the baſe, is called the *Axis* of the Cone.

6.

If the Axis be perpendicular to the Plane of the baſe; the Cone is called a *right Cone*: but if the axis inclines to it, the Cone is then call'd a *Scalene Cone.*

7.

If a Plane KDd be drawn thro' the vertex K of the Cone, parallel to the plane of any Conic-Section MAN; the indefinite right line Dd, formed by the concurrence of that plane, and the plane of the Cone's baſe, is call'd a *Directrix.*

8.

A Conic-Section MAN is call'd an *Ellipſis*, when the directrix Dd* falls without the Cone's baſe; an *Hyperbola*, when the ſame * falls within or cuts that baſe; and a *Parabola*, when it only touches * it.

* Fig. 112.
* Fig. 113.
* Fig. 114.

The interſection nam of the Hyperbolical plane, and the oppoſite Cone, is called *oppoſite Hyperbola* to the former MAN.

9.

A right line drawn in the plane of a Conic-Section, which touches only the Section in one point, and being continued both ways, does not cut nor fall within it; is called a *Tangent.*

COROLLARY I.

FIG. 112. 144. ALL the Sides of the Cone will interſect the Elliptical Plane, ſince they meet all the plane KDd parallel thereto in the point K. From whence it appears, that the Ellipſis returns into itſelf.

COROLLARY II.

FIG. 113. 145. ALL the Sides of the Cone, produced if neceſſary, will meet the Hyperbolical Plane, except only KD, Kd, which are drawn from the vertex K to the interſections of the Directrix, and the circumference of the baſe; ſince they meet all the plane DKd parallel to the former, in the point K. Moreover the ſides of the part of the Cone KdYD, do form the Hyperbolical Curve NAM, and the Sides produced of the part KdXD, do form the oppoſite Hyperbolical Curve nam. From whence it appears, that each of the oppoſite Sections are infinite, and do not return into themſelves.

COROLLARY III.

FIG. 114. 146. ALL the Sides produced, will meet the Parabolical Plane, except the Side KD, which is drawn from the vertex K thro' the point D, where the directrix touches the baſe: becauſe they meet all the Plane KDd parallel to the former in the point K. From whence it appears, that the Parabola extend itſelf, *ad infinitum.*

COROLLARY IV.

FIG. 113. 147. IT is alſo evident, from the generation of the Cone's Surface, that its interſections KD, Kd with any plane, paſſing thro' the vertex K, will be two right lines.

LEM-

LEMMA III.

148. IF *a Plane* EFGH *revolves about a line* EF *as an axis, its interſections* GH, gh, *with another plane* AB, *will always be right lines.* FIG. 107. 108.

If the axis EF *be parallel to the plane* AB, *all the interſections* GH, gh, *of thoſe planes, will be parallel to the axis* EF, *and therefore parallel between themſelves.* FIG. 107.

But if the axis EF *is not parallel to the plane* AB, *the interſections* GH, gh, *will always meet in the ſame point* G, *wherein the axis* EF *does meet the Plane* AB. FIG. 108.

This is ſo plain, that it requires no demonſtration.

LEMMA IV.

149. IF *upon a radius* CA *(produced if neceſſary) of a Circle, you take* ∺ CB : CA : CS, *and the line* DB *be drawn at right angles to* CA: *I ſay, the line* ab *joining the points of contact of two tangents, drawn from any point of the line* DB, *will always paſs thro' the point* S. FIG. 109. 110.

For joining the center C and the point of concourſe D of the tangents by a right line, meeting ab in P; then becauſe of the right angled ſimilar triangles CBD, CPS, we have CP : CS :: CB : CD, or (CS×CB) $\overline{CA}^2$ = CP×CD. Therefore &c.

LEMMA V.

150. IF *the lines* Da, Db, *touch a Circle in* a, b, *and if from their interſection* D, *a line* DY *be drawn, meeting the circumference in* X, Y, *and the line* ab *in* S; *I ſay,* DX : DY :: SX : SY. FIG. [illegible]

Thro' the points Y, X, draw the lines Yy, Xx, parallel to ab, interſecting the Curve in x, y, and the tangent in F, E; then the line drawn thro' the points y, x, will paſs thro' the point D. Therefore DE : DF :: EX : FY :: Ex : Fy, or $\overline{DE}^2 : \overline{DF}^2$ :: (XEx =) $\overline{aE}^2$: (YFy =) $\overline{aF}^2$. Alſo DE : DF :: aE : aF; and conſequently DX : DY :: SX : SY.

THE-

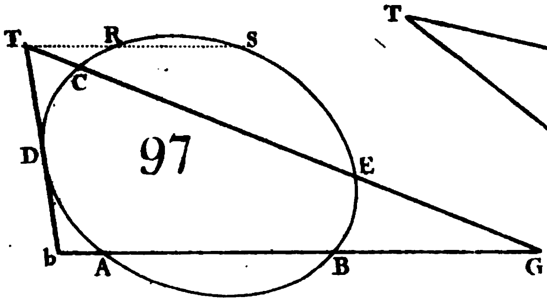
THEOREM XXIV.

Fig. 115, 116. 151. *If two right lines* MN, mn, *terminating in a Conic-Section, interſect each other in* P, *and are parallel to two other lines* KD, Kd, *given in poſition: I ſay, the Rectangles* MPN, mPn, *are always to each other in a given ratio.*

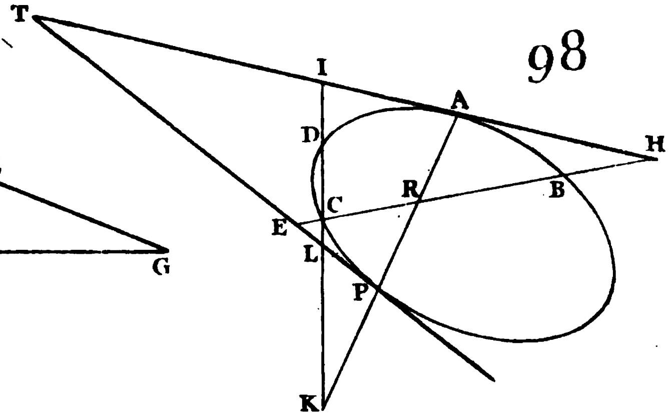
The Planes drawn thro' the parallels KD, MN, and Kd, mn, will form two right lines YXD, yxd, in the plane of the baſe, the lines KMY, KNX, and Kmy, Knx, in the Cone's Surface and their common interſection the line KPS, which meets the plane of the baſe in the point S, wherein the lines YXD, yxd, interſect each other. Now if thro' the point S you draw the line RV, in the plane YKD, parallel to MN, and the line ru, in the plane yKd, parallel to mn, and you make DK$=a$, dK$=b$, YD$=y$, DX$=x$, yd$=z$, d$x=v$. Then becauſe the triangles KPM, KSR, and KPN, KSV, are ſimilar, as are likewiſe the triangles KPm, KSr, and KPN, KSu; we ſhall have $\overline{KP}^2 : \overline{KS}^2 :: MPN : RSV :: mPn : rSu$. But becauſe of the parallel lines KD, RV, we have YS : RS :: YD (y) : DK (a), and SX : SV :: DX (x) : DK (a); and multiplying orderly YSX : RSV :: $xy : aa$, or RSV$=\frac{aa}{xy}\times YSX$, likewiſe r$Su=\frac{bb}{vz}\times ySx$. Therefore MPN : mPn :: $\frac{aa}{xy}\times YSX : \frac{bb}{vz}\times ySx$. But becauſe the lines dK (b), DK (a) as well as the points d, D are given; and by the property of the Circle the rectangles YSX, ySx, are equal; and the rectangles YDX (xy) ydx, (vz) remain always the ſame, whereſoever the lines DY, dy, are drawn; it follows, that the Rectangles MPN, mPn, are always to each other in the given ratio of $\frac{aa}{xy}$ to $\frac{bb}{vx}$.

COROLLARY I.

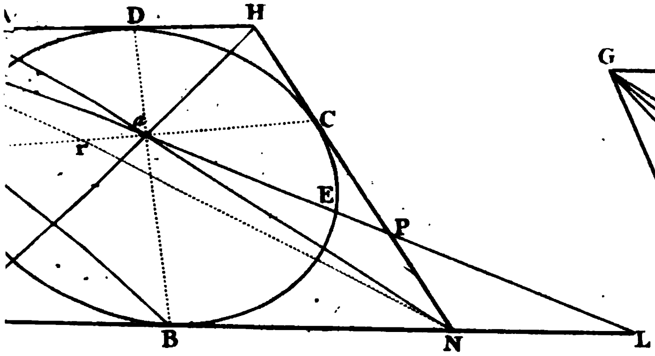
Fig. 117. 152. It is evident, that if in the Hyperbola, the line NM becomes parallel to one of the Sides of the Cone, which paſſes thro' the interſection of the directrix, and the circumference of

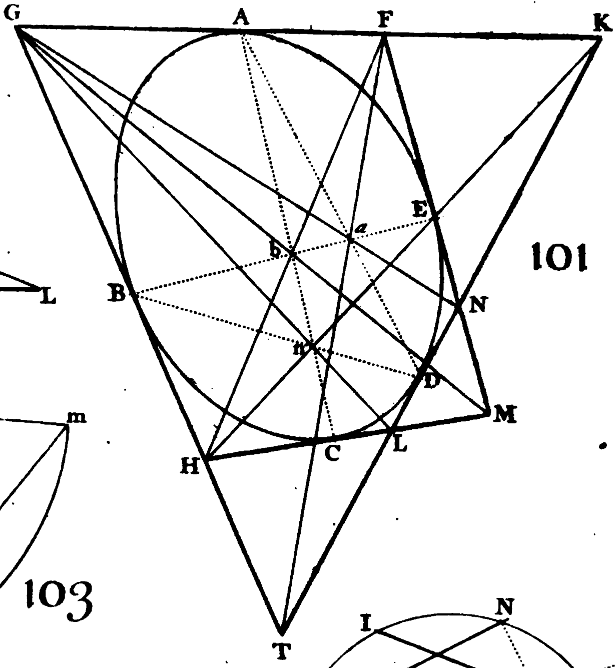


97

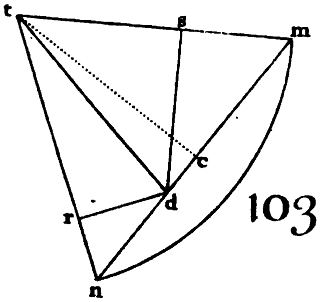


98

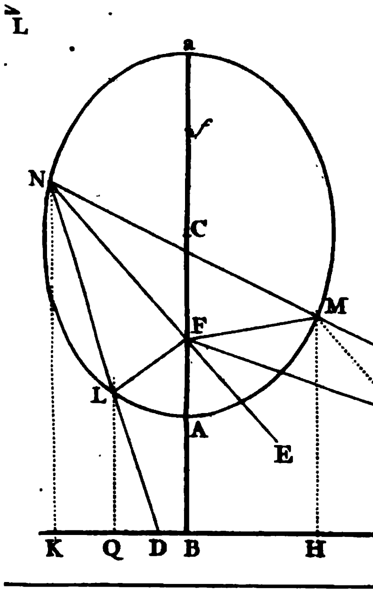




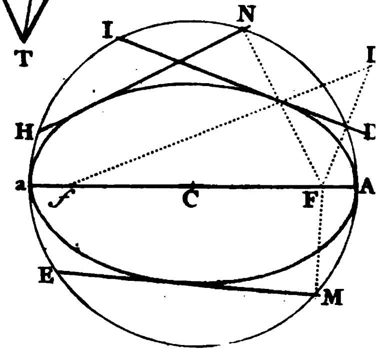
101



103



105



106

UNIV. OF MICH.

of the Cone's baſe, or in the Parabola, be parallel to the Side KX of the Cone, which paſſes thro' the point of contact X of the directrix, it will meet the Curve but in one point M; and the Rectangle mPn, will then be to MP in a given ratio.

For, if thro' the point P the line ab be drawn in the plane YKX, parallel to the line YX, and the reſt as before; we ſhall have $mPn : aPb :: rSu : YSX$; or becauſe * $rSu = \frac{bb}{vz} \times ySx$, and * Art. 151.
$YSX = ySx$, $mPn : aPb :: \frac{bb}{vz} \times ySx : YSX :: bb : vz$. But the line Pb is given, ſince MP is parallel to KX, and aP is always as MP; Therefore $mPn : MPb :: bb : vz$.

COROLLARY II.

153. IF the plane $ymKdx$ be ſuppoſed to revolve about the line Kd as an axis; it is evident, that all its interſec- Fig. 115, ſections mn with the plane MmN parallel to the axis Kd 116, 117. of revolution, will be always parallel to each other; and all its interſections dy with the plane of the Cone's baſe, will always paſs thro' the point d. And when the plane yKd touches the Cone only at the Sides KY, KX, the interſection mn will become the tangents Me, Nf.

COROLLARY III.

154. HEnce if the Plane YKX paſſes thro' the points of contact Y, X, of the tangents drawn from the point d to the Cone's baſe, the line mn will be always biſected in P by the line MP, whereſoever the plane yKd paſſes. For * $yS : yd ::$ * Art. 150.
$xS : xd$; and ſince * $yS : yd :: rS : Kd$, and $xS :: xd : Su : Kd$; * Art. 151.
rS will be $= Su$, and becauſe ru, and mn are parallel to each other; it follows that $mP = Pn$.

COROLLARY IV.

155. IN the Ellipſis and Hyperbola, If the angular point d of Fig. 115, the plane yKd, moves along the directrix Dd; it is 116.

evident that the line Kd will remain always parallel to the plane MmN of the Section, and the line YX, joining the points of contact of the tangents drawn from the point d to the baſe of the Cone, will always paſs thro' a given point*. Therefore the plane YKX will turn about a line as an axis, which therefore will always * biſect MN; ſince DX : DY :: SX : SY. * *Art.* 149. * *Art.* 144.

COROLLARY V.

* *Art.* 154. FIG. 115, 116, 117.

156. BEcauſe the line MP biſects always * the parallel lines mn: it follows that the line MP is a diameter. Whence it appears: That all the diameter of an Ellipſis or Hyperbola, biſect each other * in one point: and that in the Parabola they are all parallel to the Side KX of the Cone, which paſſes thro' the point of contact CX of the directrix dX, and conſequently are parallel between themſelves. * *Art.* 155.

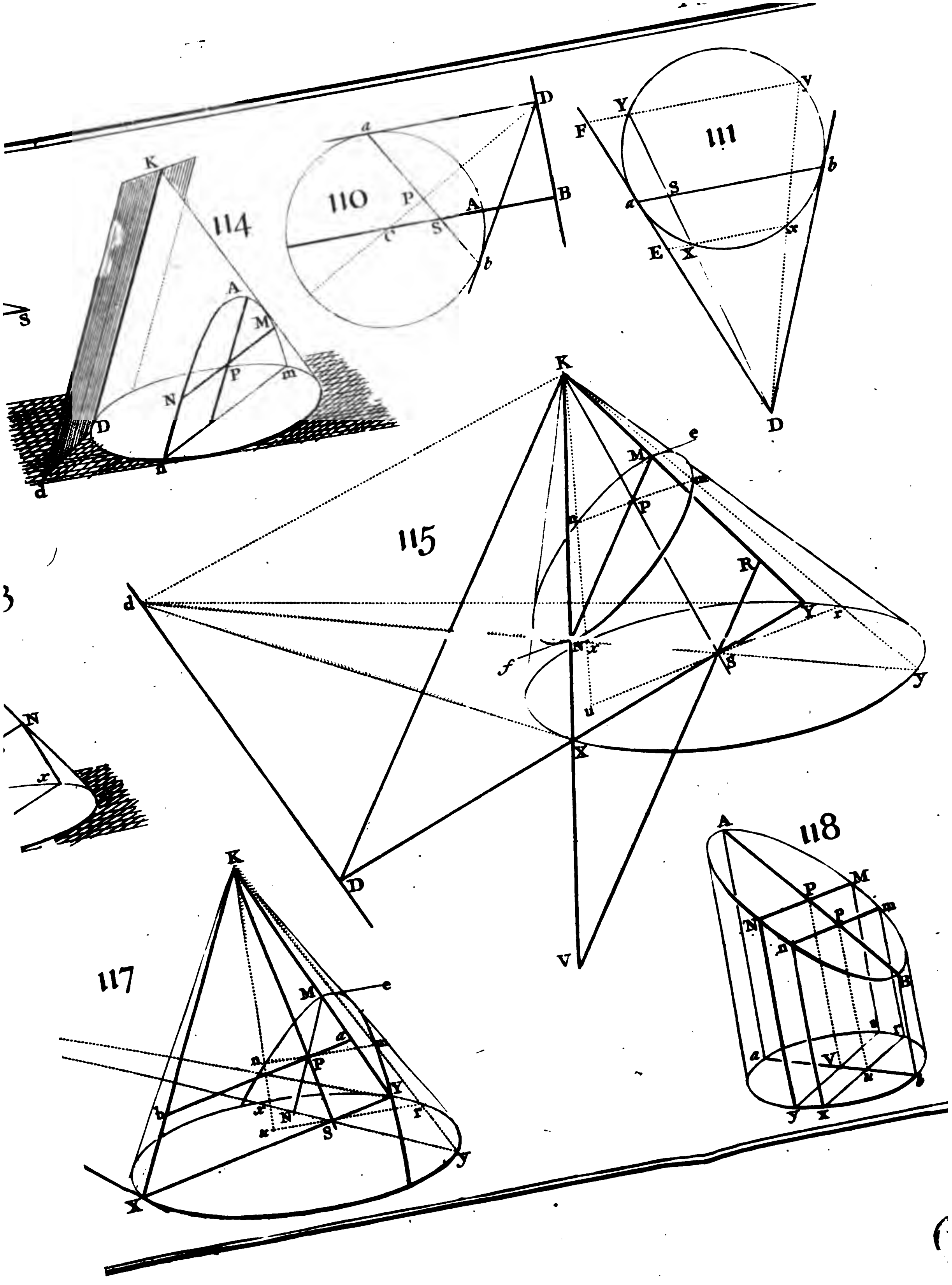
THEOREM XXV.

FIG. 118.

157. I*F a right Cylinder* Ab, *be cut by a plane* BMAN *obliquely to its baſe* asb, *the Section* BMAN *will be an Ellipſis; that is, if any two parallel lines* MN, mn, *terminating in the Section, are cut by another line* AB *terminating likewiſe in the Section, in* P, p: *I ſay* APB : ApB :: MPN : mpn.

For the Planes drawn thro' thoſe lines perpendicular to the baſe of the Cylinder, will form in its Surface the parallel lines A*a*, Bb, Ms, mr, N*y*, n*x*, and in the baſe the parallel lines r*x*, s*y*, and the line *a*b, and their interſections will form the lines p*u*, PV. Now becauſe of thoſe parallel lines, we have 1.° *a*V : *au* :: AP : Ap; 2.° bV : b*u* :: BP : Bp; and multiplying orderderly, *a*Vb : *au*b :: APB : ApB. In the ſame manner may be proved that sV*y* : *rux* :: MPN : mpn. But by the property of the Circle *a*Vb = sV*y*, and *au*b = *rux*. Therefore APB : ApB :: MPN : mpn.

The End of the firſt Book.



114
K
A
M
N
P
m
D
d
n
S
110
a
D
P
A
B
C
S
b
111
Y
V
F
S
b
a
E
X
x
D
115
K
e
M
m
n
P
R
Y
r
d
N
x
f
S
y
u
X
D
V
N
x
117
K
M
e
d
m
n
P
Y
b
x
N
r
u
S
y
X
118
A
P
M
p
m
N
n
B
s
r
a
V
u
b
y
x

BOOK II.

Of FLUXIONS.

SECTION I.

LEMMA I.

158. THE product of a^4 multiply'd into a^2 is $a^{4+2} = a^6$, that of $\overline{a+x}^3$ into $\overline{a+x}^2$ is $\overline{a+x}^{3+2} = \overline{a+x}^5$. In general the product of a^m multiply'd into a^n is a^{m+n}, that of $\overline{a+x}^m$ into $\overline{a+u}^n$ is $\overline{a+x}^{m+n}$. Likewiſe the product of $x^m \times \overline{a+x}^m$ multiplyed into $x^n \times \overline{a+x}^n$ is $x^{m+n} \times \overline{a+x}^{m+n}$. This is evident from the Compoſition of Powers.

LEMMA II

159. THE quotient of a^4 divided by a^2 is $a^{4-2} = a^2$; that of $\overline{a+x}^6$ divided by $\overline{a+x}^3$ is $\overline{a+x}^{6-3} = \overline{a+x}^3$. In general the quotient of a^m divided by a^n is a^{m-n}, that of $\overline{a+x}^m$ by $\overline{a+x}^n$ is $\overline{a+x}^{m-n}$. Likewiſe the quotient of $x^m \times \overline{a+x}^m$ divided by $x^n \times \overline{a+x}^n$ is $x^{m-n} \times \overline{a+x}^{m-n}$. This is the inverſe of the firſt Lemma.

COROLLARY I.

160. BEcauſe $\frac{a^m}{a^n} = a^{m-n}$ and $\frac{\overline{a+x}^m}{\overline{a+x}^n} = \overline{a+x}^{m-n}$; it follows that the denominator of a fraction or a broken number may be put into the numerator, by changing only the Sign of its exponent. For if we ſuppoſe $a^m = 1$, the fraction $\frac{a^m}{a^n} = a^{m-n}$, will become $\frac{1}{a^m} = a^{-n}$. Likewiſe if $\overline{a+x}^m = 1$, the fraction $\frac{\overline{a+x}^m}{\overline{a+x}^n} = \overline{a+x}^{m-n}$ will become $\frac{1}{\overline{a+x}^n} = \overline{a+x}^{-n}$.

COROLLARY II.

161. HEnce, will $\sqrt{x} = x^{\frac{1}{2}}$, $\sqrt[3]{x} = x^{\frac{1}{3}}$, $\sqrt{a+x} = \overline{a+x}^{\frac{1}{2}}$, and in general $\sqrt[m]{a+x} = \overline{a+x}^{\frac{1}{m}}$. For if both ſides of the Equation $\sqrt[m]{a+x} = \overline{a+x}^{\frac{1}{m}}$ are elevated to the Power m we ſhall have $a+x = \overline{a+x}^{\frac{1}{m}+\frac{1}{m}+}$, &c. continued to m terms, that is $= \overline{a+x}^{\frac{m}{m}} = a+x$.

LEMMA III.

162. IF a Series $a^n + ba^{n-1} + bba^{n-2} + b^3a^{n-3}$ &c. (formed by a binomial $a+b$, of which the exponent n of the firſt number decreaſes, and the exponent of the ſecond increaſes by Unity) be multiply'd by the difference $a-b$, the product will be $= a^{n+1} - b^{n+1}$. For the ſaid product will be $a^{n+1} + ba^{n-1} - ba^{n-1} + bba^{n-2} - bba^{n-2} + \text{- - - -} b^{n+1}$, which ſhews that all the terms vaniſh except the firſt and the laſt.

COR.

COROLLARY III.

163. HEnce if a binomial $a^n - b^n$ be divided by the difference $a-b$ of the two numbers, the quotient will be compoſed of as many poſitive terms as the exponent n contains units, and the greateſt exponent will be $n-1$; therefore $a^3 - b^3$ divided by $a-b$, gives $aa+ab+bb$.

LEMMA IV.

164. *SUppoſe a Point* (*a*) *to be continually moved by two forces at the ſame time, the one conſtant or given, and in the direction* AP, *and the other varying every where according to ſome given law, and in the direction* AE *parallel to the ſemi-ordinate* PM, *ſo that the point* (*a*) *may deſcribe a Curve-line* AMn; *then the* Velocity *with which the point* (*a*) *arrives at any given place* M, *in the direction* AP, *will be to the* Velocity *with which it arrives at* M *in the direction* AE, *as the ſubtangent* TP *is to the ſemi-ordinate* PM. FIG. 119.

Let the point *a* continue to move uniformly with the Velocity with which it arrived at M. It is evident, that inſtead of continuing to deſcribe a Curve-line Mn, it will deſcribe a right line MN touching the Curve in M. For if the motion in the direction AE be a retarded motion, ſo that the Curve be concave next to AP; and if the lines Lp, Nq are any where drawn on both Sides of the point M, parallel to AE, meeting the Curve in m, n, the line MN in L, N, and the line EM parallel to AP in F, r; then, whilſt the point *a* moves from F to M, and from M to r, in the direction AP, with an uniform motion, it will move from m to F and from r to n in the direction AE, with the retarded motion: and ſince the velocity of the point *a* is leſs in M than in m, and greater in M than in n, by ſuppoſition; the ſpace mF, will be greater, and nr leſs than the ſpaces LF Nr, which the point *a* would deſcribe uniformly in the ſame time with the velocity it has at M. Therefore the line TMN, lies on the ſame ſide of the Curve, and meets the ſame but in one point M.

Now becauſe, whilſt the point *a* moving uniformly from M to N with the velocity with which it arrived at M in the direction M N, moves from M to r, with the velocity with which it arrived at M in the direction A P or E M, and from r to N, with the velocity with which it arrived at M in the direction A E or P M: and ſince the Spaces deſcribed uniformly in the ſame time, are as the velocities with which they are deſcribed: it follows that the velocity of the point *a* at M or P in the direction AP, is to the velocity at M in the direction AE, as Mr to rN, or becauſe of the ſimilar triangles TPM, MrN, as the Subtangent TP is to the Semi-ordinate PM. *W. W. D.*

COROLLARY I.

165. BEcauſe the point *a* deſcribes the Curve AM, with the compound motion of the two forces acting in the directions AP and AE; the velocity in any place M of the point *a*, generated by that compound motion, may be ſaid to be the velocity with which the point *a* moving in the Curve arrives at M. For it is evident, that the more or leſs the compound motion is accelerated or retarded in a given time, the point will deſcribe greater or leſſer parts of the Curve.

N. B. Though we have ſuppoſed one of the forces which move the point *a* over the Curve given, yet they both may vary according to different laws; for the Demonſtration above holds good, in all Caſes which poſſibly can be ſuppoſed with reſpect to their variation.

DEFINITIONS.

THE velocity with which the point *a* arrives at M or P in the direction E M or AP, is call'd the *Fluxion* of the Abſciſs AP.

The velocity with which the point *a* arrives at M in the direction AE or PM, is call'd the *Fluxion* of the Semi-ordinate PM.

And the velocity generated by the compound motion, and with which the point *a* arrives at M, is call'd the *Fluxion* of the Curve-line AM.

In general, if the line AP expreſſes any variable Quantity whatſoever, generated by a continued motion, varying according to any given law; the velocity at any given point P is call'd the *Fluxion* of the Line or Quantity AP.

And reciprocally, thoſe Lines or Quantities are ſaid to be the *Fluents* of their Fluxions.

N. B. Hereafter we ſhall always expreſs the conſtant or invariable Quantities by the firſt letters of the alphabet as a, b, c, d, e, &c. and the flowing or variable Quantities by the latter as, s, t, u, x, y, z, and their Fluxions by the ſame letters with a dot or point over them, as $\dot{v}$, $\dot{x}$, $\dot{y}$, $\dot{z}$.

PROBLEM I.

166. TO *find the Fluxion of* yy.

Suppoſe $x = yy$, and AM to be the Curve of that Equation, which will be the common Parabola: for if AP$=x$, PM$=y$, and the parameter of the diameter AP be unity; then will $x = yy$ and the Subtangent TP will* be $=2x = 2yy$. Therefore TP $(2yy)$: PM (y) :: $\dot{x} : \dot{y}$, or $\dot{x} = 2y\dot{y}$, conſequently $2y\dot{y}$ is the Fluxion required of yy. *W. W. D.* * *Art.* 22.

PROBLEM II.

167. TO *find the Fluxion of* y^3.

Let $x = y^3$, and AM be the Curve of that Equation. If Ap$=x$, pm$=y$, and Aq$=u$, qn$=z$; then will $u = z^3$; and ſubſtracting the firſt Equation from the laſt, we ſhall have $u - x = z^3 - y^3 = \overline{z-y} \times \overline{zz+zy+yy}$. Now if the line nm meets the diameter AP in t; then will qn$-$pm $(z-y)$: qp $(u-x)$ :: pm : pt :: 1 : $zz+zy+yy$ and if the lines pm, qn, are ſuppoſed to approach continually until they coincide with PM; it is evident, that the line nt, will likewiſe coincide with the tangent TM, and z will become $=y$. Therefore PM : PT :: 1 : $3yy$ :: $\dot{y} : \dot{x}$, or $\dot{x} = 3yy\dot{y}$; and conſequently $3yy\dot{y}$, will be the fluxion deſired of y^3. *W. W. D.*

In general, let it be required to find the Fluxion of y^m.

The ſame thing being ſuppoſed as before, we ſhall have $u - x = z^m - y^m = \overline{z-y} \times$ into $z^{m-1} + yz^{m-2} + yyz^{m-3} +$ *&c.* continued to m terms*: therefore pm : pt :: 1 : $z^{m-1} + yz^{m-2} +$ *&c.* and when the line nt becomes the tangent TM; z will become * *Art.* 163.

* *Art.* 163. come $=y$, and $z^{m-1}+yz^{m-2}+$&c.* $=my^{m-1}$. Conſequently PM : PT :: 1 : my^{m-1} :: $\dot{y}:\dot{x}$; or $\dot{x}=m\dot{y}y^{m-1}$; therefore $m\dot{y}y^{m-1}$ will be the fluxion deſired of y^m.

PROBLEM III.

168. TO *find the Fluxion of* zy.

Fig. 120. Let the Curves AmMr, AnNs have the ſame vertex A, and be ſuch that the rectangle pm×pn, of the ſemi-ordinates be always equal to the common abſciſs Ap multiplied into unity; therefore, if Ap$=x$, pm$=y$, pn$=z$, Aq$=r$, qr$=u$, qs$=v$, we ſhall have $x=yz$, $r=uv$, and $r-x=uv-yz$, or $1=\frac{u+y}{2}\times\frac{v-z}{r-x}+\frac{v+z}{2}\times\frac{u-y}{r-x}$: and if the lines rm, sn, meet the diameter

* *Art.* 167. AP in t, u; then will* $\frac{v-z}{r-x}=\frac{z}{\text{pu}}$, $\frac{u-y}{r-x}=\frac{y}{\text{pt}}$; and ſo the laſt Equation will become $\frac{uz+yz}{2\text{pu}}+\frac{vy+zy}{2\text{pt}}=1$. Now when the ordinates mn, rs, coincide with NM, the lines rt, su, will likewiſe coincide with the tangents MT, NV: and y will become $=u$, $z=v$, pu$=$PV, pt$=$PT. Conſequently the laſt E-

* *Art.* 164. quation will become $\frac{yz}{\text{PV}}+\frac{zy}{\text{PT}}=1$, or $y\dot{z}+z\dot{y}=\dot{x}$ becauſe* $\frac{z}{\text{PV}}=\frac{\dot{z}}{\dot{x}}$ and $\frac{y}{\text{PT}}=\frac{\dot{y}}{\dot{x}}$. *W. W. D.*

COROLLARY I.

169. BEcauſe we have always Nr : rM :: MP : TP. It is manifeſt that the ratio's of the vaniſhing lines nr, rM, nM, are equal to the ratio's of the Fluxions of the ſemi-ordinate PM, abſciſs AP, and the Curve AM; ſince rn, rN, become equal in this caſe as well as Mn and MN. Therefore we ſhall hereafter expreſs indifferently thoſe ratio's, either by the vaniſhing lines nr, rM, nM, or by the Sides of the triangle MPT.

COR.

COROLLARY II.

170. THE Fluxion of xyz, will be $\dot{x}yz+\dot{y}xz+\dot{z}xy$. For if $xy=u$, (whoſe fluxion is $\dot{x}y+\dot{y}x=\dot{u}$); then will $xyz=uz$: therefore the fluxion of uz, $=\dot{u}z+\dot{z}u=\dot{x}yz+\dot{y}xz+\dot{z}xy$ (becauſe $\dot{x}y+\dot{y}x=\dot{u}$, and $xy=u$) will be the fluxion of xyz deſired.

COROLLARY III.

171. HEnce the Fluxion of z^{1}, is ${}_{n}\,\dot{z}z^{\frac{1}{n}-1}$. For if $z^{\frac{1}{n}}=x$, or $z=x^{n}$ the fluxion of which is $\dot{z}=n\dot{x}x^{n-1}$, or $\dot{z}z^{\frac{1}{n}}=n\dot{x}x^{n}$, becauſe $z^{\frac{1}{n}}=x$; thereſore ſubſtituting z for x^{n}, and dividing by nz; we ſhall have $\frac{1}{n}\dot{z}z^{\frac{1}{n}-1}=\dot{x}$, which is the fluxion ſought.

COROLLARY IV.

172. THE Fluxion of $\overline{a+x}^{n}$, will be $n\dot{x}\times\overline{a+x}^{n-1}$. For ſuppoſe $\overline{a+x}^{n}=z$, or $a+x=z^{\frac{1}{n}}$; then will $\dot{x}=\frac{1}{n}z^{\frac{1}{n}-1}\dot{z}$, or $n\dot{x}\times\overline{a+x}^{n}=z^{\frac{1}{n}}\dot{z}$; therefore $n\dot{x}\times\overline{a+x}^{n-1}=\dot{z}$; becauſe $\overline{a+x}^{n}=z$, and $a+x=z^{\frac{1}{n}}$.

In the ſame manner may be proved that the fluxion of $\overline{a+x}^{\frac{1}{n}}$, is $\frac{1}{n}\dot{x}\times\overline{a+x}^{\frac{1}{n}-1}$.

COROLLARY V.

173. THE Fluxion of $x^{m}\times\overline{a+x}^{n}$, is $m\dot{x}x^{m-1}\times\overline{a+x}^{n}+n\dot{x}x^{m}\times\overline{a+x}^{n-1}$. For if $\overline{a+x}^{n}=z$, then will $x^{m}\times\overline{a+x}^{n}$

$=x^m z$, and the fluxion of $x^m z$ will be, $m\dot{x}zx^{m-1}+\dot{z}x^m=m\dot{x}x^{m-1}\times\overline{a+x}^{\,n}+n\dot{x}x^m\times\overline{a+x}^{\,n-1}$, becauſe $\overline{a+x}^{\,n}=z$, and $n\dot{x}\times\overline{a+x}^{\,n-1}=*\dot{z}$. Hence we diſcover the following;

* Art. 172.

General Rules for finding the Fluxions of variable Quantities.

I.

174. TO find the Fluxion of any quantity as az^m, where the exponent m expreſſes any finite nnmber whole or broken.

* Art. 167. *Multiply the quantity by the * exponent of the variable quantity, and change one of its dimenſions into its fluxion. Thus* $ma\dot{z}z^{m-1}$, *will be the Fluxion required.*

II.

175. TO find the Fluxion of two variable quantities $ay^m z^n$ elevated to any povers m, n, and multiplyed into each other.

* Art. 168. *Multiply the fluxion of each quantity into * the other quantity, the Sum of the two products* $az^n\times m\dot{y}y^{m-1}+ay^m\times n\dot{z}z^{n-1}$ *will be the Fluxion required.*

III.

176. TO find the Fluxion of a binomial $\overline{a+x^n}^{\,m}$ elevated to any power m.

Multiply the product of the fluxion of the variable term x^n *and the exponent* m *of the binomial, into the binomial, whoſe exponent leſſen by unity, that is ** $mn\dot{x}x^{n-1}\times\overline{a+x^n}^{\,m-1}$ *will be the Fluxion.* * Art. 172.

IV.

177. TO find the Fluxion of any binomial multiply'd into a variable quantity elevated to any power, *viz.* $x^r\times\overline{a+x^n}^{\,m}$.

Multiply the fluxion of the quantity which multiplies the Binomial into the Binomial, multiply likewiſe the fluxion of the Binomial into that quantity: then the Sum of thoſe two products will be the Fluxion required: viz. * $r\dot{x}x^{r-1}\times\overline{a+x^{n}}^{m}+nm\dot{x}x^{n+r-1}\times\overline{a+x^{n}}^{m-1}$. * *Art.* 173.

N. B. Though we commonly ſay that ſuch a quantity, as for example, $m\dot{y}y^{m-}$, is the Fluxion of y^{m}; yet that expreſſion is not ſufficiently accurate: Therefore, the ſenſe in which we deſire to be underſtood is, that $1 : my^{m-1} :: \dot{y} : m\dot{y}y^{m-1}$ that is, unity is to my^{m-1}, or $\dot{y}$ is to $m\dot{y}y^{m-1}$, as the fluxion or velocity with which y is generated, is to the fluxion, or contemporary velocity with which y^{m} is generated, and ſo for the reſt.

SCHOLIUM.

As in the uſe of fluxions, there is required no more than the Ratio's of the Velocities, with which variable Quantities as x, y, z, &c. are generated in the ſame time, and ſince we may expreſs thoſe Ratio's, by ſuch lines as increaſe or decreaſe in the ſame proportion, as thoſe Velocities: it follows, that we may conceive the Velocities, with which thoſe lines are generated in the ſame time, as the fluxions of the former, $\dot{x}$, $\dot{y}$, $\dot{z}$, &c. or as the ſecond fluxions of x, y, z, &c. And likewiſe the velocities, with which the lines, expreſſing the Ratio's of theſe ſecond fluxions, are generated, as the fluxions of the ſecond fluxions, or as the third fluxions of x, y, z. And ſo on to ſo many ſubordinate degrees of fluxions as thoſe Ratio's vary.

The ſecond fluxions of x, y, z, are denoted by $\ddot{x}$, $\ddot{y}$, $\ddot{z}$; the third by $\dddot{x}$, $\dddot{y}$, $\dddot{z}$, &c. and ſo on.

To illuſtrate this, let the abſciſs AP, be always to the correſpondent ſemi-ordinate PM, as the degree of velocity denoted by $\dot{x}$, is to the degree of the contemporary velocity denoted by $\dot{y}$; and let the ratio of theſe fluxions $\dot{x}$, $\dot{y}$, be expreſſed by any Equation, ſo that the Curve AM may be deſcribed: then the fluxion $\ddot{x}$ of $\dot{x}$, or the ſecond fluxion of x, will be to the fluxion $\ddot{y}$ of $\dot{y}$, or the ſecond fluxion of y, as the ſubtangent TP is to the ſemi- FIG. 119.

ordinate PM. Therefore, if $\dot{x}=\dot{y}^3$, TP will be $=3\dot{x}=3\dot{y}^3$, and conſequently TP : PM :: $3\dot{y}^3 : \dot{y} :: \ddot{x} : \ddot{y}$, or $\ddot{x} = 3\dot{y}\dot{y}\ddot{y}$.

COROLLARY I.

178. FROM whence it eaſily appears, that the ſecond flux-ions are the variations of the velocities, with whi Quantities are generated in the ſame time; the third fluxion the variations of theſe firſt variations; and the fourth fluxion the variations of the ſecond variations, &c.

COROLLARY II.

179. IF AP is an axis, that is, if the ſemi-ordinates PM at right angles upon the diameter AP and if $\dot{x}$, denote the fluxions of the abſciſs AP, of the ſemi-ordinate PM and of the Curve AM: then becauſe theſe fluxions are as * Sides of the triangle TPM, we ſhall have $\dot{z}\dot{z}=\dot{x}\dot{x}+\dot{y}\dot{y}$; the fluxion of this Equation, will be $\dot{z}\ddot{z}=\dot{x}\ddot{x}$, when $\dot{y}$ is conſt $\dot{z}\ddot{z}=\dot{y}\ddot{y}$, when $\dot{x}$ is conſtant; and $\dot{x}\ddot{x}+\dot{y}\ddot{y}=0$, when $\dot{z}$ is con-ſtant. Which ſhews, that if one of theſe firſt Fluxions conſtant, the other two, are reciprocally as their fluxions.

* Art. 165.

It is evident, that this Corollary may be alſo apply'd to right angled Triangle. For if one of the Sides is conſtant, other two will be always reciprocally as their Fluxions.

COROLLARY III.

180. THE Method of finding Fluxions being known, T gents may be drawn. to any Curve whoſe Nature given. For, becauſe * $\dot{y}:\dot{x}::\text{PM}(y):\text{TP}=\frac{y\dot{x}}{\dot{y}}$, if inſtead the fluxions $\dot{x}, \dot{y}$, their ratio be ſubſtituted, which will be fou by the help of the Equation of the Curve, the Subtangent m be determined. And if AP is an axis, and MK a perpendicu to the tangent; the right angled ſimilar triangles TPM, MP

will give $PK=\frac{y\dot{y}}{\dot{x}}$. Therefore, the ſubperpendicular PK, may be found likewiſe.

N. B. We may obſerve, that if the fluxion $\dot{y}$ of the ſemi-ordinate, be ſuppoſed equal to the ſemi-ordinate y itſelf, in the Equation of a Curve; then the fluxion $\dot{x}$ of the abſciſs, will be equal to the Subtangent TP; ſince $TP=\frac{y\dot{x}}{\dot{y}}$ becomes $=\dot{x}$. Therefore, if the Fluxion of the abſciſs is taken as uſual, and the terms in which the ſemi-ordinate is, are only multiplyed by the Exponent of that quantity; the fluxion of the abſciſs, will be equal to the Subtangent. Example, let $x=y^m$ be the Equation of the Curve: then will $TP=\dot{x}=my^m$. Likewiſe, if $axy+y^3=x^3+bbx$ be the Equation; we ſhall have $axy+a\dot{x}y+3y^3=3xx\dot{x}+bb\dot{x}$, and $TP=\frac{axy+3y^3}{3xx+bb-ay}$.

SECTION II.

The Uſe of Fluxions in finding Maximums *and* Minimums.

DEFINITION.

LET MEm be a Curve whoſe ordinates are parallel to each other: and let this Curve be ſuch, that while the abſciſs AP continually increaſes, the ſemi-ordinate PM increaſes alſo, until it comes to a certain point D, and afterwards decreaſes, or on the contrary, if the ſame decreaſes until it comes to a certain point D, and afterwards increaſes. Fig. 121. 122.

The line DE is called the greateſt or leaſt ſemi-ordinate, or a *Maximum* or *Minimum*.

COROLLARY.

Fig. 121, 123. 181. HENCE, if the ſemi-ordinate PM to the axis AB, conſiſts of one or more variable quantities as AP; it is evident, 1°. that if while AP continually increaſes, PM begins from nothing to increaſe likewiſe continually without decreaſing, or begins from infinity to decreaſe continually without increaſing; the expreſſion will contain neither a *Maximum* nor a *Minimum.* 2°. If PM begins from nothing to increaſe continually for a while, and afterwards decreaſes continually without increaſing; the Expreſſion will contain a *Maximum* DE; on the contrary,* Fig. 122, 124. if it begins from infinity to decreaſe continually for a while and afterwards increaſes continually without decreaſing; the Expreſſion will contain a *Minimum.* Fig. 125, 3°. Laſtly, * if PM begins from nothing to increaſe for a while, until it becomes DE, and afterwards decreaſes till it becomes FG, and then increaſes again, Fig. 126. or if it begins * from infinity to decreaſe until it becomes DE and afterwards increaſes till it becomes FG, and then decreaſes again and ſo on alternately: the Expreſſion will contain ſome *Maximums* and ſome *Minimums.* And in the firſt caſe, the firſt DE will be a *Maximum,* the ſecond FG a *Minimum,* the third a *Maximum* again; and in the ſecond caſe, the firſt DE will be a *Minimum,* the ſecond FG a *Maximum,* the third a *Minimum* again, and ſo alternately in both caſes, as many times as there are *Maximums* and *Minimums* in the Expreſſion.

GENERAL PROBLEM.

182. THE *Nature of the Curve* MEm, *being given, to find the greateſt or leaſt ſemi-ordinate* DE, *or which is the ſame thing, an Expreſſion being given, to find its* Maximums *and* Minimums.

It is evident, that when the Semi-ordinate PM becomes the Fig. 121. 122. greateſt or leaſt DE, the tangent TM, will either become parallel to the axis AB as in figures 121, 122, or perpendicular to the ſame as in figures 123, 124; and therefore the fluxion of the abſciſs AP will be infinite with regard to the fluxion of the ſemi-ordinate PM, in the firſt caſe, and on the contrary, it w be nothing with regard to the fluxion of the ſemi-ordinate in the ſecond. Conſequently, if the fluxion of the ſemi-ordinate PM, be ſuppoſed $=0$, you will find ſuch an Expreſſion AD for AP that the ſemi-ordinate DE be the greateſt or leaſt required in the firſt caſe; and if the fluxion of the abſciſs AP be ſuppoſed $=0$, or which is the ſame thing, if the fluxion of the ſemi-ordinate PM is ſuppoſed infinite, the Expreſſion AD found for AP will determine the greateſt or leaſt ſemi-ordinate DE in the ſecond. *W. W. D.* FIG. 123. 124.

N. B. It may be obſerved 1°. that ſuppoſing the ſemi-ordinate PM conſtant or invariable, is the ſame thing as ſuppoſing its fluxion equal to nothing. 2°. That if the Expreſſion found by taking the fluxions of the variable quantities be a fraction: then the numerator will give a value for AP, when the tangent becomes parallel to the abſciſs, and the denominator when the tangent becomes parallel to the ſemi-ordinate; ſince in the firſt caſe the fluxion of the abſciſs is infinite, and in the ſecond that of the ſemi-ordinate.

EXAMPLE I.

183. TO *divide the line* AB$=a$ *in ſuch a manner, that the product* $\overline{AD}^m \times \overline{DB}^n = x^m \times \overline{a-x}^n$, *be a* Maximum. FIG. 127.

Suppoſe the perpendicular DE$=y$, ſo that $y=x^m \times \overline{a-x}^n$, be the Equation of the Curve AEB. Then the Fluxion * $m\dot{x}x^{m-1} \times \overline{a-x}^n - n\dot{x}x^m \times \overline{a-x}^{n-1}$, being ſuppoſed $=0$, and divided by $\dot{x}x^{m-1} \times \overline{a-x}^{n-1}$, gives $ma-mx-nx=0$, or AD$=x=\frac{ma}{m+n}$. * *Art.* 177.

If n and m are any whole or broken poſitive numbers, DE will be always a *Maximum*, for if $x=0$, or $x=a$, the Equation $y=x^m \times \overline{a-x}^n$ will give $y=0$; and therefore, DE * will be a *Maximum*. And if n be negative, and leſs than m, DE will be always a *Minimum*; ſince, the Equation $x=\frac{ma}{m+n}$, gives $x>a$; and the Equation $y=x^m \times \overline{a-x}^n$ will become $y=\frac{x^m}{\overline{x-a}^n}$: and when * *Art.* 181. No. 2.

* *Art.* 181. No. 3. x is equal to a, or infinite, y will be infinite alſo ; and conſequently DE will be a *Minimum*.

In this laſt caſe, the Problem may be thus laid down. Continue out the given line AB towards B, to the point D, in ſuch a manner that $\frac{\overline{AD}^m}{\overline{DB}^n}$ be a *Minimum*.

EXAMPLE II.

Fig. 128. 184. O*F all the Rectangles* PMmp, *that may be inſcribed in a Semi-circle* AMma, *to find the greateſt.*

Let $Aa=2a$, $CP=x$; then will $2PC\times PM=y=2x\sqrt{aa-xx}$; the fluxion of which, $2\dot{x}\sqrt{aa-xx}-\frac{2xx\dot{x}}{\sqrt{aa-xx}}$, being made $=0$, will give $CP=x=\pm\sqrt{\frac{1}{2}aa}$; and y will be a *Maximum* ; ſince when $x=a$, y will be $=0$. Which ſhews that the arch AM, or am is 45 degrees.

EXAMPLE III.

185. O*F all the Cylinders that may be inſcribed in a Sphere, to find the greateſt.*

Suppoſing the reſt as before, $Pp\times\overline{PM}^2=y=2x\times\overline{aa-xx}$, will be as the Cylinder deſcribed by the rotation of the rectangle pM about the diameter Aa. Therefore, the fluxion $2aa\dot{x}-6xx\dot{x}$, being ſuppoſed $=0$, will give $CP=x=\pm\sqrt{\frac{1}{3}aa}$.

EXAMPLE IV.

186. O*F all the Cones that may be inſcribed in a Sphere, to find that whoſe convex Surface is the greateſt.*

It is evident, that $Am\times pm=\sqrt{2aa+2ax}\times\sqrt{aa-xx}$, will be as the Surface generated by the line Am revolving about the diameter Aa. Therefore the Fluxion $\frac{a\dot{x}\sqrt{aa-xx}}{\sqrt{2aa+2ax}}-\frac{x\dot{x}\sqrt{2aa+2ax}}{\sqrt{aa-xx}}$

being $=0$, gives $aa-2ax-3xx=0$, or $x=\frac{a}{3}$.

EXAMPLE V.

187. *Of all the Parallelepipedons, which are equal to a given Cube* a^3, *and have a given Side* b, *to find that having the leaſt Surface.*

Let x be one of the unknown Sides; then $\frac{a^3}{bx}$ will be the other; and the Sum of the alternate planes, $bx+\frac{a^3}{x}+\frac{a^3}{b}$, will be equal to half the Surface. Therefore the fluxion $b\dot{x}-\frac{a^3\dot{x}}{xx}$ being $=0$, gives $xx=\frac{a^3}{b}$; which is a *Minimum*, ſince the Expreſſion $bx+\frac{a^3}{x}+\frac{a^3}{b}$, cannot become $=0$, but may become infinite.

From what we have ſaid with regard to the Method of finding *Maximums* and *Minimums*, the Reader may eaſily perceive, how to apply it in any other caſe; and as in the Collection of Problems at the end of this Work, we give ſeveral uſeful Problems depending thereon, we ſhall not inſiſt any longer on it here.

SECTION III.

The Uſe of Fluxions in finding the Radii of Evolution.

LEMMA V.

188. If *the point* B, *moving in the radius* CA, *according to ſome law, deſcribes any line* BN *whilſt the point* A *deſcribes the circular arch* AM *about the center* C: *then the circular* Velo- Fig. 129.

city *of the point* A *at* M *conſidered in the direction of the tangent* TM, *or perpendicular to the radius* CM, *will be to the* Velocity *of the point* B *at* N *conſidered in the direction* uN *perpendicular to the radius* CM, *as* CM *is to* CN.

For the velocity with which the point A, deſcribing the * *Art.* 165. circular arch AM, arrives at M, is as * the tangent TM; and the velocity with which the point B, deſcribing the Curve BN, arrives at N, is as the Tangent VN; and because the velocity in the direction VN, is equivalent to the velocities in the directions uV, uN; the velocity of the point B at N in the direction perpendicular to CN, will be as uN. Therefore TM : uN :: CM : CN; that is, the velocities of the points A, B, at M, N, conſidered in the direction perpendicular to CM, are as the Radii or Diſtances CM, CN, from the centre C. *W. W. D.*

N. B. Hereafter we ſhall always conſider the velocity of a point in any place, and in the direction perpendicular to the line moving with an angular motion, as a circular velocity, that is, as the velocity with which the ſaid point would arrive at that place if moving in a circular arch.

COROLLARY

FIG. 130. 189. IF the point C did likewiſe move according to ſome law, ſo that it may deſcribe the Curve CO, whilſt the point B deſcribes the Curve BN, and the point A the circular arch AM; the circular velocity of the point A at M, will be to the circular velocity of the point B at N, with reſpect to the point or centre O, as the radius OM is to the radius ON. For if the line OX be drawn perpendicular to CM; it is evident, that the points M, N will endeavour to turn about the point O only with the differences Tt, Vu, of their circular velocities TM, VN, to the circular velocity XO of the point O. Therefore OM : ON, or Xt : Xu :: Tt : Vu.

DEFINITION.

FIG. 131. IF you conceive a Curve BDF, concave the ſame way, to be envelop'd by a thread ABDF, one end of which being fix

at F, and the other A from the poſition AB of the tangent at B, moves tightly ſo as to diſengage itſelf continually from the Curve BDF: the ſaid end or point A of the thread, will deſcribe a Curve AMN, which is called an *Involute Curve.*

The Curve BDF, is called the *Evolute* of the Curve AMN, or the Evolute Curve. And each of the ſtrait parts DM, FN, of the thread, is called the *Radius of Evolution.*

COROLLARY I.

190. Becauſe the length of the thread ABDF, remains always the ſame; it follows, that the part DF of the Evolute, is equal to the difference of the Radii DM, FN, which part from its extremities; as likewiſe DB=DM—AB. And if AB=0, the parts BD, BDF, of the Evolute, will be always equal to the correſpondent Radii.

COROLLARY II.

191. It is evident, that the Radius of Evolution, is always perpendicular to the deſcribing part of the Involute. For if the point A did continue to move uniformly, when arrived at M, with its laſt velocity, it would deſcribe a right line, touching the Circle deſcribed with the radius DM about the center D; and becauſe it would likewiſe deſcribe a tangent to the * Curve AM; it follows, that the ſaid point would proceed in a right line perpendicular to DM. * *Art.* 164.

COROLLARY III.

192. Since the ſame right line TM, may touch both the Curve AM and the Circle deſcribed with the Radius DM about the center D; it follows, that the velocity with which the point A arrives at M, deſcribing the Involute, will be equal to the uniform velocity, with which that point would deſcribe a circular arch in the ſame time with the radius DM: and therefore the Radius of Evolution DM. may be conſidered as invariable as well as the point or center D, whilſt the other variable lines vary.

COR-

COROLLARY IV.

193. BEcauſe the Curvatures of different Circles, are reci cally as their Radii; it follows, that the Curvatur M of the Involute AM, is to the Curvature at any other po as N, reciprocally as the Radius of Evolution FN is to the R dius of Evolution DM.

GENERAL PROBLEM.

194. *THE Nature of the Curve* AM *being given, to fin Length* DM *of the Radius of Evolution, drawn f a given point* M.

Caſe 1. Let all the ſemi-ordinates PM, be perpendicula the line AP, and let the part Mm of the tangent, expreſ Fluxion of the Curve AM, and Kk that of the line AK; the Kn be drawn perpendicular, and kn parallel to KM, and AP= PM $=y$, PK $=r$, KM $=c$, Kk $=q\dot{x}$; the ſimilar triangles M Knk, will give Kn $=\frac{qy\dot{x}}{c}$, for the circular velocity of the p

* *Art.* 164. 165. K. But PM : KM :: TP : TM :: * $\dot{x}$: Mm $=\frac{c\dot{x}}{y}$. Theref
* *Art.* 189. DM : DK :: $\frac{c\dot{x}}{y}:\frac{qy\dot{x}}{c}$; and by diviſion DM : KM (c) :: $\frac{c\dot{x}}{y}$ $-\frac{qy\dot{x}}{c}$; or DM $=\frac{c^3}{cc-qyy}$.

If the line DE be drawn perpendicular to MP produced; will ME $=\frac{ccy}{cc-qyy}$, and DE $=\frac{ccr}{cc-qyy}$.

Caſe 2. Let all the ſemi-ordinates PM, iſſue from the ſa

FIG. 132. point P, and draw KPT perpendicular to PM: then if Mm e preſſes the Fluxion of the Curve MN, Kn (sy) the fluxi PK, and KL perpendicular to PK, the circular velocity or ion of the point K with reſpect to the radius PK; and if KR drawn perpendicular, and nk, LR, parallel to KM; the c

re

rence kR, between KR, and Kk, will expreſs the circular velocity or fluxion of the point K with reſpect to the radius DK. For becauſe the velocity in the direction PK, is retarded, whilſt the line DM and the right angle MPK, revolve about the points D and P, when the Curve MN is concave next to the point P, and ſo the circular velocity KL, will be likewiſe retarded; and therefore, the circular velocity Kk, muſt be ſubſtracted from KR in this caſe. Now becauſe of the ſimilar triangles KPM, nkK, we ſhall have Kk$=\frac{sy\dot{y}}{c}$, and the radius PM will be to the radius PK as * the circular velocity ($\dot{x}$) of the point M, is to the * *Art.* 188. circular velocity KL$=\frac{r\dot{x}}{y}={}^{*}\dot{y}$, of the point K: likewiſe KM: * *Art.* 180. KP :: KL : KR$=\frac{r\dot{y}}{c}$, and therefore kR$=\frac{r\dot{y}-sy\dot{y}}{c}$. But KP:KM ::PM:TM:: * $\dot{y}$:Mm$=\frac{c\dot{y}}{r}$, and conſequently DM:DK::$\frac{c\dot{y}}{r}$: * *Art.* 165. $\frac{r\dot{y}-sy\dot{y}}{c}$, and by compoſition DM:KM (c) :: $\frac{c\dot{y}}{r}:\frac{c\dot{y}}{r}+\frac{r\dot{y}-sy\dot{y}}{c}$ or DM$=\frac{c^3}{cc+rr-rsy}$. If DE be drawn perpendicular to PM; then will ME$=\frac{ccy}{cc+rr-rsy}$, and DE$=\frac{ccr}{cc+rr-rsy}$.

Otherwiſe.

195. If ME$=z$, MT($\dot{v}$): then the ſimilar triangles TPM, MED, FIG. 131. 132. will give DM$=\frac{v\dot{z}}{\dot{x}}$; and becauſe the Radius of Evolution DM, may be conſidered as * invariable, by taking $\dot{x}$ as conſtant, the * *Art.* 192. Fluxion $\frac{\ddot{v}z-\dot{v}\dot{z}}{\dot{x}}$, will be $=0$. Therefore $z=-\frac{\dot{v}\dot{y}}{\ddot{v}}$, in figure 131, becauſe $\dot{y}=\dot{z}$. And in figure 132, PT ($\dot{x}$): PM ($\dot{y}$)::ME (z) : DE$=\frac{z\dot{y}}{\dot{x}}$, and the radius PM is to the radius PE, as the cir-

cular velocity ($\dot{x}$) of the point M, is to the circular velo the point E or Fluxion of DE: and becauſe the ſides o right angle DEP, are reciprocally as * their fluxions; it will $\dot{y} : \dot{x} :: \mathrm{DE}\left(\frac{z\dot{y}}{\dot{x}}\right) : \frac{z\dot{y}}{y} =$ to the Fluxion of PE; and ſo $\dot{z} = \dot{y} - \frac{z\dot{y}}{y}$. Conſequently ſubſtituting for $\dot{z}$, its value into $\ddot{v}z + \dot{v}\dot{z} = 0$, it will become ME ($z$) $= \frac{\dot{v}y\dot{y}}{\dot{y}\dot{v} - y\ddot{v}}$. *W. W. D.*

* Art. 179.

COROLLARY.

196. **B**Ecauſe, TP ($\dot{x}$) : TM (v) :: PM (y) : KM (c) or $v = \frac{c\dot{x}}{\dot{y}}$; whoſe fluxion will be $\ddot{v} = \frac{\dot{y}c\ddot{x} - c\ddot{y}\dot{x}}{\dot{y}\dot{y}}$, or if $t\dot{y} = c$, $\ddot{v} = \frac{t\dot{y}\dot{y}\ddot{x} - c\ddot{y}\dot{x}}{\dot{y}\dot{y}}$, and ſubſtituting for $\dot{v}$, $-\ddot{v}$, their values, in ME $= -\frac{\dot{v}y}{\ddot{v}}$, we ſhall find ME $= \frac{cy}{c - ty}$: And ſince $cc = rr + yy$, the fluxion of which $c\dot{c} = ct\dot{y} = r\dot{r} + y\dot{y}$, or if $\dot{r} = s\dot{y}$, will give $t = \frac{rs + y}{c}$: Therefore ME $= \frac{ccy}{rr - rsy}$, becauſe $cc - yy = rr$; and DE $= \frac{cc}{r - sy}$.

FIG. 131.

By determining the quantity, q, s or t, by the help of the Equation of the Curve, the Radius of Evolution will be determined; and ſince * $\dot{x}r = y\dot{y}$, the Equation of the Curve being given, the value of r may be found, and conſequently that of q, s, or t.

* Art. 180.

FIG. 131. As the Radius of Evolution DM has but one value, ſo has the Curve AMN but one Evolute BDF.

N. B. In the Curves, where the tangent at the vertex is perpendicular to the axis, if the point M coincides with the vertex A: it is evident, that the Radius of Evolution DM becomes = KM = to PK $= r$ in that point.

Co

EXAMPLE I.

197. LET AM be the Conic-Parabola and $2px=yy$, its Equation; whoſe fluxion $p\dot{x}=y\dot{y}$, gives $r=p=\frac{yy}{\dot{x}}$, therefore $\dot{r}=0$, and * $s=0$, and ſo DE $(=\frac{cc}{r-sy})=\frac{cc}{r}=p+2x$, becauſe $cc=2px+pp$, and $p=r$. FIG. 133. * Art. 196.

In order to find the nature of the Evolute BD; draw DF (z) perpendicular to AP: then becauſe, when the point M coincides with the vertex A, MK becomes $=\text{AB}=r=p$; and if $\text{BF}=u$, we ſhall have AP+(DE) PF $=p+3x$, $\text{AF}-\text{AB}=u=3x$, and $\text{DE}-\text{PK}=\text{FK}=2x$. Therefore $\text{PK}:\text{PM}::\text{KF}:\text{DF}=z=\frac{2x}{p}\sqrt{2px}$, or $pzz=8x^3$, and $27pzz=8u^3$, becauſe $u=3x$.

EXAMPLE II.

198. LET MN be an equilateral Hyperbola, and $1=xy$, its Equation with reſpect to the Aſymptotes; whoſe Fluxion $\dot{x}y+\dot{y}x=0$, gives $-y^3=\frac{yy}{\dot{x}}=r$, and $-3yy\dot{y}=\dot{r}=$ *$s\dot{y}$, or $-3yy=s$. Therefore DE $(\frac{cc}{r-sy})=\frac{cc}{-2r}$ becauſe $-y^3=r$. Conſequently, if $\text{PF}=\frac{1}{2}\text{KT}$, the perpendicular FD to CP, will determine the Radius of Evolution required: Since ∹ PK : KM : KT: The negative Value $\frac{cc}{-2r}$ of DE, ſhews that the point D falls on the contrary ſide of the point M with reſpect to the point K. FIG. 1[illegible] * Art. 196.

EXAMPLE III.

199. LET $x=y^m$ be the Equation, which is that of all kinds of Parabola's, when m is poſitive, or of all kinds of Hyperbola's, when m is negative; whoſe Fluxion $\dot{x}=m\dot{y}y^{m-1}$, FIG. 135, 134.

gives

* Art. 196. gives $r = \frac{1}{n}y^{2-m}$, and ſo $\frac{2-m}{m}\dot{y}y^{1-m} = \dot{r} = {}^{*}s\dot{y}$, or $\frac{2-m}{m}y^{1-m} =$

Therefore DE $\left(\frac{cc}{r-sy}\right) = \frac{cc}{\overline{m-1}\,r}$, becauſe $r = \frac{1}{n}y^{2-m}$. Conſequent

if PF $= \frac{1}{m-1}$ KT, the perpendicular FD to PK, will determi

the Radius of Evolution DM required.

EXAMPLE IV.

200. LET the Curve AM be an Ellipſis or Hyperbola, wh Fig. 135, 136.
Equation is $\pm\frac{a}{p}yy = aa - uu$; then will PK=
* Art. 19. * Art. 194. $^{*}\frac{pu}{a}$, and CK $= u \mp \frac{pu}{a}$, and ſo * $q\dot{u} = u \mp \frac{p\dot{u}}{a}$, or $q = 1 \mp \frac{p}{a}$

Therefore DM $\left(\frac{c^3}{cc-qyy} = \frac{c^3}{cc-yy\pm\frac{pyy}{a}} = \frac{c^3}{pp}\right.$, becauſe cc−

$= rr = \frac{ppuu}{aa}$, and $\pm\frac{p}{a}yy = pp - \frac{pp}{aa}uu$. Conſequently, if DM
taken a fourth continual proportional to the ſemi-parameter
and KM (c); the point D will be in the Evolute.

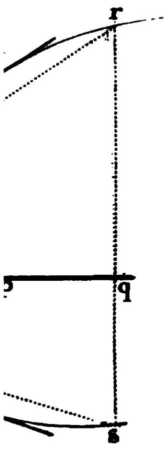
When the point M coincides with the vertex A, DM will b
come $=$ AB $= r = \frac{pu}{a} = p$, becauſe u then becomes $= a$; and w
in the Ellipſis, the point M coincides with the vertex b of the
* Art. 21. cond axis, DM will become* $= \frac{ay}{p} = \frac{ab}{p}$.

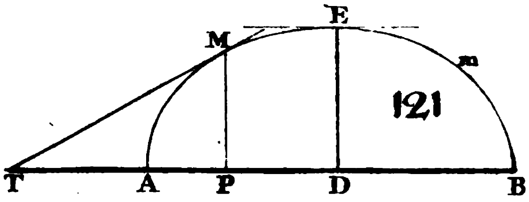
EXAMPLE V.

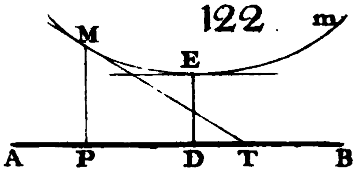
Fig. 137, 201. LET AM be the Logarithmical Curve, whoſe prope
is ſuch, that drawing any perpendicular PM (
its aſymptote FK, and a tangent MT, the ſubtangent PT
* Art. 196. will be always given. Therefore $r = \frac{yy}{a}$, and * $s = \frac{2y}{a}$, and D

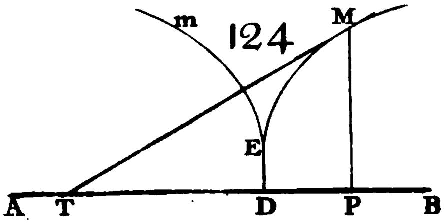
$\left(\frac{cc}{r-sy}\right) = \frac{cc}{-r}$, becauſe $r = \frac{yy}{a}$

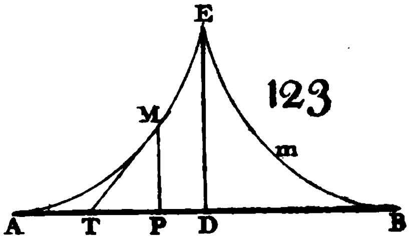
E

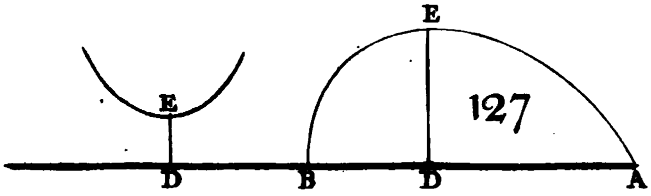


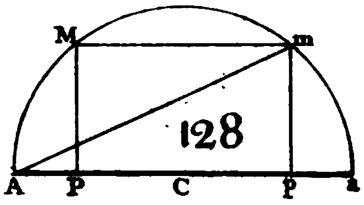

121

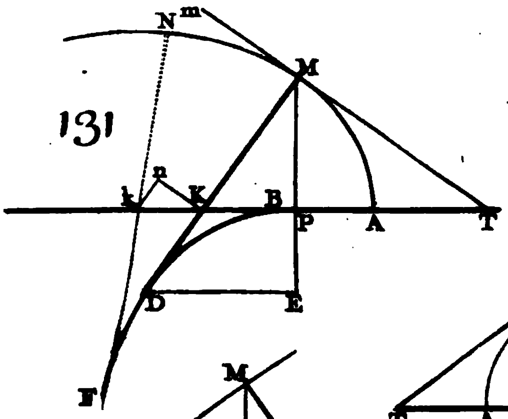

122

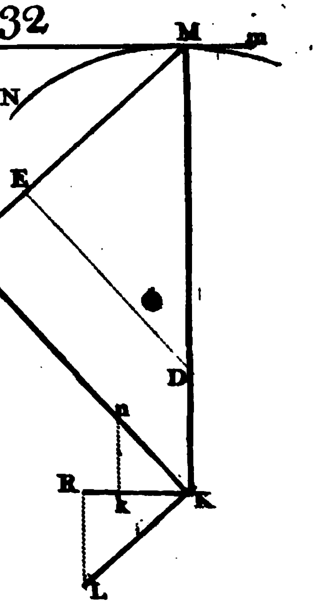

124

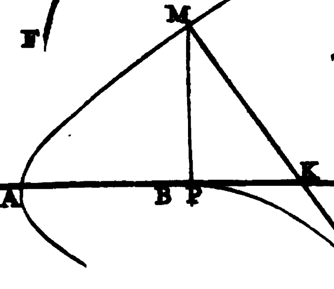

123

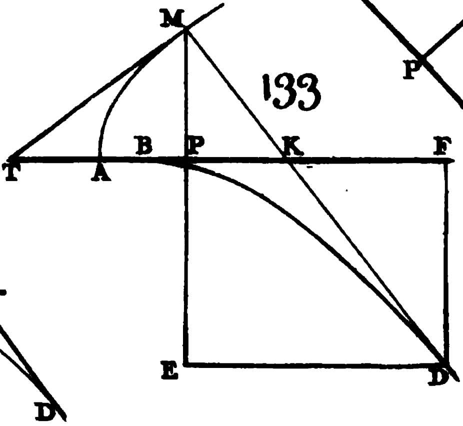

127


128


131


132


6


133

EXAMPLE VI.

202. LET PFM be the Logarithmical Spiral, whoſe property is ſuch, that the angle TMP, made by the tangent and ſemi-ordinate, is always given. Fig. 138.

Through the Pole P draw TK perpendicular to PM: then becauſe the angles of the right angled triangle MPK, are given, the ratio of the ſides PM, PK, will be given alſo. Therefore, if PK$=r=my$, we ſhall have $\dot{r}=s\dot{y}=m\dot{y}$, or $s=m$; and DM $(\frac{c^3}{cc+rr-rsy})=c=$MK.

EXAMPLE VII.

203. LET PFM be one Spiral of an infinite number formed in the circular Sector PANB, and of ſuch a nature, that the radius $\overline{PN}^m$ (1) be to its part $\overline{PM}^m$ (y^m) terminated by the Spiral, as the whole arch ANB (a) is to the part AN (z) terminated by that radius. Fig. 139.

Therefore $ay^m=z$ will be its Equation; and becauſe the radius PN is to the radius PM, as * the circular velocity $\dot{z}$ of the point N is to the circular velocity $\dot{x}$ of the point M; we ſhall have $\dot{z}=\frac{\dot{x}}{y}$; and ſubſtituting for $\dot{z}$ its value, into the fluxion $ma\dot{y}y^{m-1}=\dot{z}$, of the Equation $ay^m=z$, it will become $ma\dot{y}y^m=\dot{x}$. Therefore PK$=r=\frac{1}{ma}xy^{1-m}$, and ſo $\frac{1-m}{ma}\dot{y}y^{-m}=\dot{r}=s\dot{y}$, or $s=\frac{1-m}{ma}y^{-m}=\frac{\overline{1-mr}}{y}$, and ME $(\frac{ccy}{cc+rr-rsy})=\frac{ccy}{yy+\overline{m+1}rr}$, becauſe $cc=rr+yy$. Conſequently, if TP$+\overline{m+1}$PK : TK :: PM : ME; the perpendicular ED to PM, will determine the Radius of Evolution DM required.

* Art. 188.

EXAMPLE VIII.

FIG. 140. 204. LET PFMB, be the reciprocal Spiral, with ſuch a Property, that the Product of any arch AN (z), of a Circle deſcribed with the pole P as a center, and terminated by the radius PN and the firſt radius PA, multiplied into the part PM (y) of that radius terminated by the Spiral, be always given.

Let the arch AB, terminated by the firſt radius PA and by the Spiral, be $=n$, and PN $=a$; then will $an=yz$; and becauſe the radius PN is to the radius PM, as the circular velocity $-\dot{z}$ of the point N, is to * the circular velocity $\dot{x}$ of the point M; we ſhall have $-y\dot{z}=a\dot{x}$; and ſubſtituting for $-\dot{z}$, its value into the Fluxion $\frac{an\dot{y}}{yy}=\dot{z}$. of the Equation $an=yz$, it will become $n\dot{y}=y\dot{x}$; and ſo $r=\frac{yy}{n}$, and $\dot{r}=\frac{2y\dot{y}}{n}=s\dot{y}$, or $\frac{2y}{n}=s$. Therefore ME $\left(\frac{ccy}{cc+rr-rsy}\right)=\frac{cc}{y}$, becauſe $r=\frac{yy}{n}$ and $cc-rr=yy$. * Art. 188.

SECTION IV.

The Uſe of Fluxions in finding of Cauſticks by Refraction and by Reflection

DEFINITION.

FIG. 141, 142. IF the directions of an infinite number of Rays FA, FM, all iſſuing from the ſame luminous Point F, be altered after their concurrence with a Curve AMN, either nearer or farther from the perpendicular DM to it; and if the law of alteration be continually ſuch, that the Sine DE of the angle of incidence DME, be to the Sine DB, of the angle of refraction DMB, in the given ratio of m to n: the Curve KLH, which touches all the broken or refracted Rays, or their continuations AH, ML, is call'd the *Cauſtick* by *Refraction.*

Corollary

205 If the Cauſtick L H, be taken as an Evolute, and the Involute A Q be deſcribed from the beginning at A: then if from the center F, the circular arch AP be deſcribed, the line QL *plus* the part LH of the Cauſtick, will be always *= AH: and if the part Mm, of the tangent at M expreſſes the Fluxion of the Curve AM, and M*f*, M*g*, are drawn perpendicular, and m*f*, m*g* parallel to the incident and refracted Rays FM, ML; *f*m will expreſs the * Fluxion of PM (FM—FA) and m*g* the Fluxion of QM (AH—MLH). Now becauſe of the ſimilar triangles M*f*m, MED, and M*g*m, MBD, we ſhall have *f*m: DE:: Mm: DM; and *g*m: DB:: Mm: DM; *ex æquo*, *f*m: *g*m:: DE: DB:: *m*: *n*. Therefore the fluent PM of the fluxion *f*m, will be to the fluent QM of the fluxion *g*m, as *m* is to *n*, or $\frac{n}{m}$ PM = QM = AH—LH—LM. Hence we get HL = AH—LM—$\frac{n}{m}$ PM, Fig. 141. * Art. 190. * Art. 164.

There are ſeveral Caſes according as the incident Ray FA is greater or leſs than FM, and the refracted Ray AH, wraps about or unwraps itſelf from the part HL of the Cauſtick; but it may be proved as already, that the Difference between the incident Rays, is always to the Difference of the refracted Rays *plus* the part of the Cauſtick between thoſe Rays, as *m* to *n*. Fig. 142.

If the luminous Point F removes at an infinite diſtance from the Curve AM, that is, if the incident Rays FA, FM, become parallel; we ſhall ſtill have LH = AH—ML—$\frac{n}{m}$ PM

GENERAL PROBLEM.

206 *The Nature of the Curve AM and the luminous Point F being given, to determine the* Length ML *of the refracted Ray drawn from a given point.* Fig. 141.

Let DM be the Radius of Evolution, and $ME = a$, $MB = b$, $FM = y$, $Mf = \dot{x}$; then becauſe of the Similar triangles, ME : ML :: Mf : $Mg = \frac{b\dot{x}}{a}$; and the radius FM will be to the radius FE $(y+a)$, as the circular velocity Mf of the point M is to the circular velocity $Ee = \frac{y\dot{x}+\dot{x}a}{y}$, or fluxion of DE, which is to the fluxion Bb of DB, as m to n, that is $Bb = \frac{ny\dot{x}+na\dot{x}}{my}$. But the radius LM, is to the radius LB, as the circular velocity Mg of the point M is to the circular velocity Bb of the point B, and by diviſion LM : BM (b) :: $\frac{b\dot{x}}{a} : \frac{b\dot{x}}{a} - \frac{ny\dot{x}-na\dot{x}}{my}$, or $LM = \frac{bbmy}{bmy-any-aan}$ · which gives the following Conſtruction.

FIG. 143 Make the angle EDC equal to the angle BDM, and in the incident Ray FM, take $MK = \frac{aa}{y}$; then if C K : CE :: MB : ML, the point L will be in the Cauſtick by refraction. For by reaſon of the Similar triangles, DB : DE :: n : m :: MB : $CE = \frac{bm}{n}$; and ſo $CE - ME = CM = \frac{bm-an}{n}$, and $CM - MK = CK = \frac{bmy-any-aan}{ny}$. Therefore CK : CE :: MB : $LM = \frac{bbmy}{bmy-any-aan}$

When CK$\left(\frac{bmy-any-aan}{ny}\right)$ is negative, LM will be ſo likewiſe. Whence it follows, that when the point M falls between the points B, L, the point C will fall between the points K, E.

If the luminous Point F be next to E, or, which is the ſame, if the Curve AM be concave next to F; then y will be changed from poſitive to negative, and conſequently $LM = \frac{bbmy}{bmy-any+aan}$ in this caſe, and the Conſtruction will be the ſame.

If y be ſuppoſed to be infinite, that is, if the incident Rays

FM, FA, are parallel between themſelves; then will LM$=\frac{bmy}{bmy-any}$, becauſe *aan*$=0$, with reſpect to *bmy*, *any*. And ſince MK $\left(\frac{aa}{y}\right)$ then vaniſhes, we need only make CM : CE :: MB : LM.

If the Radius of Evolution DM be always infinite, that is, if the Curve AM becomes a right line; then ME (a) and MB (b) will be likewiſe infinite, and LM$=\frac{bbmy}{aan}$. Therefore if CK$=\frac{aa}{y}$ the reſt of the Conſtruction will be the ſame as before.

COROLLARY.

207. IF the point E falls on the other Side of DM with reſpect to the point B, and DE$=$DB: then will $m=n$, and LM$=\frac{ay}{2y\mp a}$, becauſe a is here negative and$=b$: And the Curve LH is call'd the *Cauſtick* by *Reflection* in this Caſe. FIG. 142.

If FM (y) be infinite, that is, if the incident Rays are parallel between themſelves; then will LM$=\frac{1}{2}a=\frac{1}{2}$ME: becauſe a is $=0$, with reſpect to y. It is evident, that if any two of the points F, D, L, are given, the third will be given alſo.

EXAMPLE I.

208. LET the Curve AM be the common Parabola, and the luminous Point F its focus; then it is manifeſt, that all the reflected Rays LM, will be parallel to the axis AF, therefore LM$=\frac{ay}{2y-a}$, will be infinite, and ſo $a=2y$. Conſequently, if ME$=2$MF, the perpendicular ED to ME, will determine the Radius of Evolution MD. FIG. 144.

EXAMPLE II.

209. LET the Curve AM be an Ellipſis, and the luminous Point F one of its foci; then all the reflected FIG. 145.

Rays LM will paſs thro' the other focus f. Therefore, if $Mf = z = \frac{ay}{2y-a}$, we ſhall have $ME = a = \frac{2yz}{y+z}$. But if the Curve AM be an Hyberbola, and the luminous Point F one of its foci: then the reflected Rays LM, produced at the other Side
FIG. 146. of the Curve, will all paſs thro' the other focus f, and therefore z will be negative, that is $-z = \frac{ay}{2y-a}$, or ME $(a) = \frac{2yz}{y-z}$. Conſequently, if in both Caſes, ME be taken a fourth proportional
Art. 11. to half the firſt axis * Aa $\left(\frac{y \pm z}{2}\right)$ and to the incident and reflected Rays, the perpendicular ED to ME, will determine the Radius of Evolution DM.

PROBLEM.

FIG. 147. 210. *THE Curve HL and the luminous Point F being given, to find a Curve AM, whereof HL may be the Cauſtick by Refraction.*

From the luminous Point F draw a tangent FH to HL, in which take the point A as one of the Curve ſought, and in any refracted Ray as LM, take $LK = \frac{n}{m}FA + AH - LH$, and join the points F, K: then if $Km = n$, $mf = m$; the line FM, drawn parallel to fm, will determine the point M of the Curve required. For Km (n): mf (m) :: KM: FM, or $KM = \frac{n}{m}FM$, and $(KM + LM =) \frac{n}{m}FM + LM (= LK) = \frac{n}{m}FA + AH - LH$, by Suppoſition. Therefore $\left(\frac{n}{m}\overline{FM - FA} =\right) \frac{n}{m}PM = AH - LH -$
* Art. 205. LM. Conſequently the Curve * AM thus deſcribed, will be that required. *W. W. D.*

FIG. 148, If the incident Rays FM, are parallel between themſelves; take $LK = AH - LH$, $Km = n$, and from the point m, draw mf parallel to AH and $= m$; then if the line drawn thro' the points K, f, interſects the line AP perpendicular to AH in P, the interſection of the line PM parallel to AH, and the refracted Ray LK, will be a point of the Curve required. For. $mf\ (m) : Km\ (n) :: PM : \frac{n}{m}PM = KM$, and $(KM + LM = LK =)\ \frac{n}{m}PM + LM = AH - LH$, or $\frac{n}{m}PM = AH - LH - LM$.

COROLLARY.

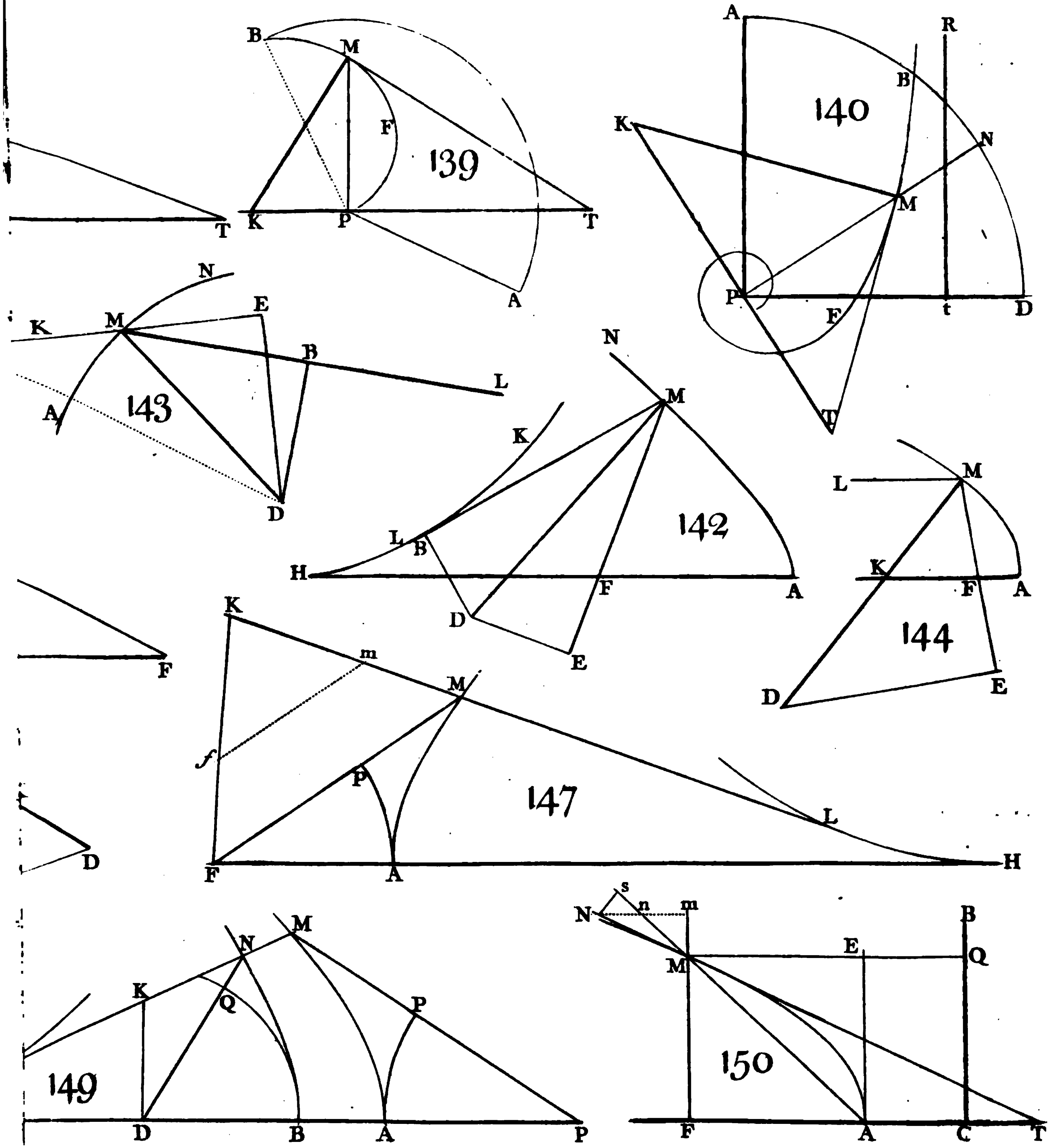
211. IT is evident, that the Curve AM changes, according as the point A is taken farther from or nearer to the luminous Point F, becauſe the value of the line LK changes. Therefore the Curve HL may be the Cauſtick by refraction of an infinite number of Curves. Fig. 147. 148.

PROBLEM.

212. *A Curve AM and the luminous Point F being given, to find another Curve BN, which ſhall refract all the Rays MN, ſo as to paſs thro' a given Point D.* Fig. 149.

It is evident, that if the point D be conſidered as the luminous Point of the Curve BN, which refracts the incident Rays DN, ſo as to coincide with the refracted Rays LM of the Curve AM; then the Curve HL will be the Cauſtick by refraction to both; and if from the centers F, D, the circular arches AP, BQ, are deſcribed, we ſhall have $\frac{n}{m}PM = AH - LH - LM$, and $\frac{n}{m}QN = BH - LH - LN$; and ſubſtracting the laſt Equation from the firſt, their difference will be $\frac{n}{m}PM - \frac{n}{m}QN = AB - MN$, becauſe $AH - BH = AB$, and $LM - LN = MN$. Therefore, if the point B be taken as one of the Curve ſought, and MK be $= \frac{n}{m}PM + \frac{n}{m}BD - AB + 2MN$: then if the point N be found, as in the preceding Problem, ſuch that $NK = \frac{n}{m}ND$; it will be one of the Curve required. For (NK+MN=MK) $\frac{n}{m}ND + MN = \frac{n}{m}PM + \frac{n}{m}BD - AB + 2MN$, or $\frac{n}{m}PM - \frac{n}{m}QN = AB - MN$, ſince $ND - BD = QN$. *W. W. D.*

If the incident Rays FM of the given Curve AM, are parallel between themſelves, the Conſtruction will be the ſame: and if the incident Rays DN of the Curve ſought, are parallel to each other; then if $MK = \frac{n}{m}PM - AB + 2MN$, and $NK = \frac{n}{m}QN$, the Point N will be in the Curve required. For (NK+MN=MK) $\frac{n}{m}QN + MN = \frac{n}{m}PM - AB + 2MN$, or $\frac{n}{m}PM - \frac{n}{m}QN = AB - MN$.


T
B
M
F
139
K
P
T
A
A
R
140
B
K
N
M
P
F
t
D
T
N
K
M
E
B
L
A
143
D
N
M
K
142
L
B
H
F
A
D
E
L
M
K
F
A
144
D
E
K
m
M
f
P
147
L
F
A
H
D
N
s
n
m
B
E
Q
M
M
N
K
Q
P
149
150
D
B
A
P
F
A
C
T

BOOK III.

Of FLUENTS.

SECTION I.

THE Method of finding Fluents, is the inverſe of that of finding Fluxions. For the fluent of $a\dot{z}z^m$, is $\frac{a}{m+1}z^{m+1}$; that of $d\dot{z}z^{n-1}\times\overline{e+fz^n}^{m}$ is $\frac{d}{fn\times\overline{m+1}}\times\overline{e+fz^n}^{m+1}$; that of $mz^n\dot{y}y^{m-1}$ $+ny^m\dot{z}z^{n-1}$, is z^ny^m; and that of $\frac{mz\dot{y}y^{m-1}-ny^m\dot{z}}{z^{n+1}}$, is $\frac{y^m}{z^n}$. Since the fluxions of theſe fluents*, are the very ſame as thoſe propoſed. From whence we deduce the following

* Art. 174. 175. 176.

General Rules for finding FLUENTS.

I.

213. OF an Expreſſion as $a\dot{z}z^m$, the exponent m being any number.

Inſtead of the fluxion ($\dot{z}$) *add unity to the exponent* (m) *and divide the Expreſſion by the Exponent* ($m+1$) *thus increaſed: then the quotient* $\frac{a}{m+1}z^{m+1}$ *will be the fluent required.*

II.

214. OF an Expreſſion as $d\dot{z}z^{n-1}\times\overline{e+fz^n}^{\,m}$, in which the quantity $d\dot{z}z^{n-1}$ without the *Vinculum*, is to the fluxion of the quantity under the *Vinculum*, in a given ratio.

Add unity to the Exponent m of the Vinculum, *and divide the Expreſſion by the product of that Exponent* $(m+1)$ *thus increaſed, multiplied by the Fluxion of the quantity under the* Vinculum $(n\dot{z}z^{n-1})$: *then the quotient* $\frac{d}{fn\times\overline{m+1}}\times\overline{e+fz^n}^{\,m+1}$ will be the fluent required.

III.

215. OF an Expreſſion as $mz^n\dot{y}y^{m-1}+ny^m\dot{z}z^{n-1}$, containing two variable quantities, z^n, y^m, each of them multiplied by the fluxion of the other, and affected with the ſame Signs; the exponents n, m, may expreſs any numbers.

* *Art.* 213. *Find the fluent of each variable quantity*, conſidering the other which multiplies the fluxion, as conſtant: then the Sum divided by the number of Terms, will be the fluent required*; viz. $\frac{z^ny^m+z^ny^m}{2}=z^ny^m$.

EXAMPLE.

* *Art.* 214. 216. HEnce, the fluent of $\dot{x}\sqrt{x}=\dot{x}x^{\frac{1}{2}}$, is * $\frac{2}{3}x^{\frac{1}{2}+1}=\frac{2}{3}x\sqrt{x}$; that of $\frac{\dot{x}}{\sqrt{x}}=\dot{x}x^{-\frac{1}{2}}$, is $2x^{-\frac{1}{2}+1}=2\sqrt{x}$; that of $x\dot{x}\sqrt{aa+xx}=x\dot{x}\times\overline{aa+xx}^{\frac{1}{2}}$, is $\frac{1}{3}\times\overline{aa+xx}^{\frac{3}{2}}$; that of $\frac{-x\dot{x}}{\sqrt{aa-xx}}=-x\dot{x}\times\overline{aa-xx}^{-\frac{1}{2}}$, is $\sqrt{aa-xx}$; that of $x\dot{x}\sqrt[3]{aa+xx}=x\dot{x}\times\overline{aa+xx}^{\frac{1}{3}}$, is $\frac{3}{8}\times\overline{aa+xx}^{\frac{4}{3}}$
Art. 215. and that of $aa\dot{x}-ax\dot{x}-xx\dot{x}$, is * $aax-\frac{1}{2}axx-\frac{1}{3}x^3$.

PROBLEM I.

217. *Suppoſe m to be any poſitive or negative broken number, to find the Caſes wherein the Fluent of an Expreſſion as $d\dot{z}z^r \times \overline{e+fz^n}^{m}$, may be found in a finite number of terms.*

Caſe 1. If we ſuppoſe $\overline{e+fz^n}^{m} = v^m$, or $e+fz^n = v$; then will $z^n = \frac{v-e}{f}$; and extracting the n root $z = \frac{\overline{v-e}^{\frac{1}{n}}}{f^{\frac{1}{n}}}$, whoſe fluxion is $\dot{z} = \frac{1}{n}\dot{v} \times \frac{\overline{v-e}^{\frac{1}{n}-1}}{f^{\frac{1}{n}}}$; and elevating $z = \frac{\overline{v-e}^{\frac{1}{n}}}{f^{\frac{1}{n}}}$ to the r power, we ſhall have $z^r = \frac{\overline{v-e}^{\frac{r}{n}}}{f^{\frac{r}{n}}}$. Now ſubſtituting the values thus found of $\dot{z}$, z^r, $\overline{e+fz^n}^{m}$ into the former Expreſſion, it will become $\frac{d}{n}\dot{v}v^m \times \frac{\overline{v-e}^{\frac{r+1}{n}-1}}{f^{\frac{r+1}{n}}}$. Therefore when $\frac{r+1}{n}-1$ is a whole poſitive number, the fluent of this laſt fluxion, equal to the fluxion propoſed, will be found in $\frac{r+1}{n}$ terms. For by the nature of powers, the binomial $v-e$ elevated to the power $\frac{r+1}{n}-1$ will contain as many terms *plus* one, as that Exponent contains units.

Caſe 2. Let $\frac{e+fz^n}{z^n} = v$; then will $e+fz^n = vz^n$, or $z^n = \frac{e}{v-f}$; the n root of which will be $z = \frac{e^{\frac{1}{n}}}{\overline{v-f}^{\frac{1}{n}}}$, whoſe fluxion is,

$\dot z = -\frac{1}{n}\dot v \times \overline{\frac{e^{\frac{1}{n}}}{v-f}}^{\frac{1}{n}+1}$; and elevating $z = \overline{\frac{e^{\frac{1}{n}}}{v-f}}^{\frac{1}{n}}$, to the r power, it will be $z^r = \overline{\frac{e^{\frac{r}{n}}}{v-f}}^{\frac{r}{n}}$; and if we add e to each Side of the Equation, $fz^n = \frac{e}{v-f}$ it will become $e+fz^n = \frac{ev}{v-f}$, or $\overline{e+fz^n}^m = \overline{\frac{e^m v^m}{v-f}}^m$. Therefore if the values of $\dot z$, z^r, $\overline{e+fz^n}^m$, are ſubſtituted in the given Expreſſion, we ſhall have $-\frac{d}{n}\dot v v^m \times \frac{e^{\frac{r+1+nm}{n}}}{\overline{v-f}^{\frac{r+1+nm}{n}+1}}$. Conſequently when $\frac{r+1+nm}{n}+1$ is a whole negative number, the binomial $v-f$ will come into the numerator, and ſo the fluent will be found in $\frac{r+1+nm}{n}$ terms.

COROLLARY

218. IF $\frac{r+1}{n}=s$, or $r=sn-1$, the Expreſſion $d\dot z z^r \times \overline{e+fz^n}^m$ will become $d\dot z z^{sn-1} \times \overline{e+fz^n}^m$ therefore when s is a whole poſitive number, the fluent of that Expreſſion may be found in a finite number of terms; for if $e+fz^n=v$, it may be changed into this other, $\frac{d}{n}\dot v v^m \times \frac{\overline{v-e}^{s-1}}{f^s}$, which being elevated to $s-1$ power, its fluent will be found by rule firſt in s terms. And if $\frac{r+1+nm}{n}=t$, or $r=tn-nm-1$; the Expreſſion $d\dot z z^r \times \overline{e+fz^n}^m$ will become $d\dot z z^{tn-nm-1} \times \overline{e+fz^n}^m$ the fluent of which may be

found, when t is a whole negative number; Since if $\frac{e+fz^n}{z^n}=v$, that expreſſion may be changed into this other, $-\frac{d}{n}\dot{v}v^m\times\overline{\frac{et}{v-f}}^{\,t+1}$; whoſe fluent will be found in t terms, as will appear by the following examples.

N. B. When s is a whole negative, or t a whole poſitive number; the fluent of $d\dot{z}z^r\times\overline{e+fz^n}^{\,m}$, may be reduced to the Quadrature of the Conic-Sections, as will be ſhewn hereafter.

EXAMPLE I.

219. LET $\dot{x}x^3\sqrt{aa+xx}$, be a fluxion propoſed: then will $d=1$, $n=2$, $3=sn-1$, or $s=2$. Therefore, if $aa+xx=v$, that Expreſſion may be changed into this other $\frac{1}{2}\dot{v}v^{\frac{1}{2}}\times\overline{v-aa}$, whoſe fluent will be * $\frac{1}{5}\overline{v-\frac{1}{3}aa}\times v^{\frac{3}{2}}$, or $\frac{3xx-2aa}{15}\times\overline{aa+xx}^{\frac{3}{2}}$ because $aa+xx=v$. * Art. 213.

EXAMPLE II.

220. LET $\dot{x}x^{-4}\sqrt{aa+xx}$, be a fluxion propoſed; then will $d=1$, $m=\frac{1}{2}$, $n=2$ and $-4=tn-nm-1$, or $-1=t$. Therefore, if $\frac{aa+xx}{xx}=v$; that Expreſſion will be changed into this other $-\frac{\dot{v}v^{\frac{1}{2}}}{2aa}$, whoſe fluent will be $-\frac{v^{\frac{3}{2}}}{3aa}=-\frac{x^{-3}}{3aa}\times\overline{aa+xx}^{\frac{3}{2}}$

N.B. It may be proper here to obſerve, that becauſe the invariable quantities, have no fluxions, it is ſome times required to add to, or ſubſtract from, the Fluents found by the preceding rules, ſome conſtant quantity, in order to make them compleat; but as there is no general rule for determining that quantity, we refer the reader to the Examples hereafter.

PROBLEM II.

221. TO elevate the Binomial $1+z$ to any power m poſitive or negative, whole or broken.

Let $\overline{1+z}^{m}=1+az+bzz+cz^3+\&c.$ the fluxion of which divided by $\dot{z}$, will give $\frac{m\times\overline{1+z}^{m}}{1+z}=a+2bz+3czz+\&c$: this laſt Equation being divided by the firſt, we ſhall have $\frac{m}{1+z}=\frac{a+2bz+3czz+\&c.}{1+az+bzz+\&c.}$:And each ſide of this Equation being reduced into an infinite Series by a continual diviſion, gives $m-mz+mzz-\&c.=a+\overline{2b-aa}z+\overline{3c-3ab+a^3}\times zz+\&c.$ Now if we compare the terms of this equation, in which z is elevated to the ſame power; we ſhall have the following Equations, $m=a$, $2b-aa=-m$, $3c-3ab+a^3=m$; which gives $b=\frac{m}{1}\times\frac{m-1}{2}$, $3c=3ab-a^3+m=\frac{m^3-3mm+2m}{2}$, or $c=\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}$. Conſequently, ſubſtituting inſtead of a, b, c, their values into the firſt Equation, it will become $\overline{1+z}^{m}=1+\frac{m}{1}z+\frac{m}{1}\times\frac{m-1}{2}zz+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}z^3+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}z^4+\&c.$ *W. W. D.*

COROLLARY I.

222. HEnce, if the Binomial, inſtead of being $\overline{1+z}^{m}$, was $\overline{e+fz^n}^{m}$, we ſhall find $\overline{e+fz^n}^{m}=e^m+\frac{m}{1}fz^ne^{m-1}+\frac{m}{1}\times\frac{m-1}{2}ffz^{2n}e^{m-2}+\&c.$ Or if $A=\frac{1}{e}$, $B=A\frac{m}{e}$, $C=B\frac{m-1}{2}$, $D=C\frac{m-2}{3}$

$E = D\frac{m-3}{4e}$, we ſhall have $\overline{e+fz^n}^{\,m} = e^m \times$ into $1 + Amfz^n + B\frac{m-1}{2}ffz^{2n} + C\frac{m-2}{3}f^3z^{3n} +$, &c.

COROLLARY II.

223. HEnce, we have likewiſe a general *Form*, for elevating any Multinomial, as $a + by + cyy +$ &c. to any Power *m*, in a very conciſe manner. For by ſuppoſing its firſt Term *a* equal to the firſt Term *e*, of the Binomial $\overline{e+fz^n}^{\,m}$, and the reſt $by + cyy + dy^3 +$ &c. equal to the ſecond fz^n; and ſubſtituting for fz^n and its Powers, $by + cyy + dy^3 +$ &c. and its Powers, into the Series e^m xinto $1 + Amfz^n + B\frac{m-1}{2}ffz^{2n} + C\frac{m-2}{3}f^3z^{3n} +$ &c. the above general Form will be accommodated for elevating a Multinomial to any Power *m*. The Reaſon of which is manifeſt; for, if inſtead of the ſecond Term *r*, of a Binomial $1+r$, elevated to any whole Power, (ſuppoſe to the Square, $1+2r+rr$) you ſubſtitute any Number of Quantities $a+b+c+$ &c. you will have that Power of the Quantities $1+a+b,+$ &c. and as the ſame Rule ſerves* for elevating a Binomial to any power; ſo will this be true in any other Caſe, as will appear by the following Examples.

* *Art.* 221.

EXAMPLE I.

224. To find the ſquare Root of $aa+bx+xx$. Suppoſe $m=\frac{1}{2}$, $e=aa$, $fz^n = bx+xx$; Then will $A=\frac{1}{aa}$, $B=\frac{A}{2aa}$, $C=-\frac{B}{4aa}$, $D=-\frac{3C}{6aa}$, $E=-\frac{5D}{8aa}$, &c. and $a \times$ into $1+\frac{1}{2}A \times \overline{bx+xx} - \frac{1}{4}B \times \overline{bx+xx}^2 - \frac{3}{6}C \times \overline{bx+xx}^3 -$ &c. will be the Root required.

EXAMPLE II.

225. TO reduce the fraction $\frac{1}{aa+bx+xx} = \overline{aa+bx+xx}^{-1}$ into an infinite Series. Suppoſe $m = -1$, $e = aa$, $fz^n = bx+xx$, then will $A = \frac{1}{aa}$, $B = -\frac{A}{aa}$, $C = -\frac{B}{aa}$, $D = -\frac{C}{aa}$, $E = -\frac{D}{aa}$, &c. and $\frac{1}{aa} \times$ into $1 - A \times \overline{bx+xx} - B \times \overline{bx+xx}^2 - C \times \overline{bx+xx}^3 -$, &c. will be the Series required.

SCHOLIUM.

BY the properties of Powers, if the coefficient of the firſt term of a Binomial elevated to any whole power be unity; that of the ſecond will expreſs the Sum of the Roots; that of the third, the Sum of all the Rectangles made of thoſe Roots; that of the fourth, the Sum of all the Parallelepipedons; that of the fifth, the Sum of all the Products of the Roots taken by four and four at a time, and ſo on. Therefore, if m Number of different Things as a, b, c, d, &c. *are conſidered as thoſe Roots; then* $\frac{m}{1} \times \frac{m-1}{2}$, *will expreſs the Number of times, that thoſe Things may be taken by* 2 *and* 2 *at a time;* $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$, *the Number of times they may be taken by* 3 *and* 3; $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, *the Number of times they may be taken by* 4 *and* 4; *and ſo on. For Inſtance, if* $m = 6$; *then will* $\frac{m}{1} \times \frac{m-1}{2} = 15$, *be the Number of times that* 6 *Things may be taken by* 2 *and* 2 *at a time differently; if* $m = 8$, *then will* $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} = 56$, *be the Number of times that* 8 *Things may be taken by* 3 *and* 3 *differently; and if* $m = 4$; *then will*

will $\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}=4$, *be the Number of times that* 4 *Things a, b, c, d, may be taken by* 3 *and* 3 *differently,* viz. *abc, abd, acd, bcd. From hence may likewiſe be deduced in a very ſhort manner, that* $m\times\overline{m-1}$, *expreſſes the Number of different Poſition, of m different Things a, b, c,* &c. *taken by* 2 *and* 2; *and* $m\times\overline{m-1}\times\overline{m-2}$, *the Number of different Poſitions of m different Things, taken by* 3 *and* 3; *and* $m\times\overline{m-1}\times\overline{m-2}\times\overline{m-3}\times$ &c. *continued to m Factors, the Number of different Poſitions of m different Things taken m and m. For Example, if* $m=2$; *then will* $m\times\overline{m-1}=2$, *be the Number of different Poſitions of two different Things, as a, b, taken by* 2 *and* 2, viz. *ab, ba*: *if* $m=3$; *then will* $m\times\overline{m-1}=6$, *be the Number of different Poſitions of three different Things, a, b, c, taken by* 2 *and* 2, viz. *ab, ac, bc, ba, ca, cb, and ſo on.*

PROBLEM. III.

226. TO *expreſs the* Fluent *of a Binomial, as* $d\dot{z}z^{r-1}\times\overline{e+fz^n}^m$, *by an infinite Series.*

Becauſe $\overline{e+fz^n}^m = e^m\times$ into $1+Amfz^n+B\frac{m-1}{2}ffz^{2n}+C\frac{m-2}{3}f^3z^{3n}+$&c, if this Series be multiplied by $d\dot{z}z^{r-1}$, and the fluent of each term be found*, we ſhall have $de^m z^r\times$ into $\frac{1}{r}+A\frac{m}{r+n}fz^n+\frac{1}{2}B\frac{m-1}{r+2n}ffz^{2n}+\frac{1}{3}C\frac{m-2}{r+3n}f^3z^{3n}+$ &c. for the fluent required; and $A=\frac{1}{e}$, $B=A\frac{m}{e}$, $C=B\frac{m-1}{2e}$, $D=C\frac{m-2}{3e}$, $E=D\frac{m-3}{4e}$, &c.

The Law of this Series being known, it may be continued at pleaſure to any Number of Terms, and may ſerve as a general Form for all Binomial Expreſſions, as will appear by the following Examples,

* *Art.* 213.

EXAMPLE I.

227. LET $\dot{x}\sqrt{x-xx}$, or $\dot{x}x^{\frac{1}{2}}\times\overline{1-x}^{\frac{1}{2}}$, be the Fluxion propoſed: Then if we ſuppoſe $z=x$, $r-1=\frac{1}{2}$, $d=1$, $m=\frac{1}{2}$, $-f=1$, $n=1$, $e=1$; we ſhall have $d\dot{z}z^{r-1}\times\overline{e+fz^{n}}^{m}=\dot{x}x^{\frac{1}{2}}\times\overline{1-x}^{\frac{1}{2}}$. Therefore $A=1$, $B=\frac{1}{2}A$, $C=-\frac{1}{4}B$, $D=-\frac{3}{6}C$, $E=-\frac{5}{8}D$, &c. and $x^{\frac{1}{2}}\times$ into $\frac{2}{3}-\frac{1}{1.5}Ax-\frac{1}{2.7}Bxx+\frac{3}{3.9}Cx^{3}-\frac{5}{4.11}Dx^{4}+$ &c. will be the fluent required.

EXAMPLE II.

228. LET $\frac{\dot{y}}{\sqrt{1-yy}}$ or $\dot{y}\times\overline{1-yy}^{-\frac{1}{2}}$, be the fluxion propoſed. Then if we ſuppoſe $y=z$, $r=1$, $-1=f$, $-\frac{1}{2}=m$, $n=2$, $d=1$, $e=1$; we ſhall have $d\dot{z}z^{r-1}\times\overline{e+fz^{n}}^{m}=\dot{y}\times\overline{1-yy}^{\frac{1}{2}}$, and $y\times$ into $1+A\frac{1}{2.3}yy-\frac{1}{2}B\frac{3}{2.5}y^{4}+\frac{1}{3}C\frac{5}{2.7}y^{6}-\frac{1}{4}D\frac{7}{2.9}y^{8}+$ &c; for the fluent required; and $A=1$, $B=-\frac{1}{2}A$, $C=-\frac{1}{4}B$, $D=-\frac{5}{6}C$, $E=-\frac{7}{8}D$, &c. or ſubſtituting for A, B, C, D, &c. their values; the fluent will become this, $y+\frac{1}{2.3}y^{3}+\frac{1.3}{2.4.5}y^{5}+\frac{1.3.5}{2.4.6.7}y^{7}+\frac{1.3.5.7}{2.4.6.8.9}y^{9}$, &c.

N. B. It is to be obſerved 1.°, that the Denominators of the Terms of the Series, muſt always be greater than their Numerators; otherwiſe the Series will diverge, and therefore not approach nearer and nearer, to the true Value of the fluent required, 2.°, that the Law of the Series is here always known, ſo that when for the greater accuracy, more Terms than we have given, are required; theſe Series may be continued at pleaſure.

SECT.

SECTION II.

The Uſe of Fluxions in finding the Expreſſions of Areas, Surfaces and Solids, together with the Rectification of Curve-lines.

GENERAL PROBLEM.

229. TO *find the* Fluxions *of Areas, Surfaces and Solids.*

Fig. 150.

Suppoſe that the Semi-ordinate PM, perpendicular to the Axis AP, or the Surface deſcribed by that Semi-ordinate in the rotation of the figure AMP about the Axis AP, moves from the beginning at A along the line AP, with a continued Motion according to ſome Law; then the fluxion of the figure generated by that Motion, will be equal to the product of the generating quantity PM, multiplied by the Fluxion of the Abſciſs AP. For the degree of Velocity with which any part of that Figure is generated, in a given time, will be always proportional, to the Sum of the Products of each Particle of the generating quantity, multiplied by its Velocity in that Place; and becauſe the Velocities of each of the particles, are here equal by Suppoſition, that is, equal to the fluxion of the abſciſs AP; it follows that the Product of the whole generating quantity PM multiplied by the common Velocity or fluxion of AP, will be the Fluxion required.

In the ſame manner may be proved, that the Fluxion of the Convex Surface of the Solid, is equal to the Product of the generating circumference (whoſe Radius is PM) multiplied by the fluxion of the arch AM. For if you imagine another Curvilinear figure whoſe correſpondent abſciſſes and ſemi-ordinates, are always equal to the arch AM and the generating circumference: then the conſequence will be, that the Area of that figure, will be always equal to the Surface deſcribed; and therefore its fluxion (which is equal to the Product of the ſemi-ordinate multiplied by the fluxion of the Abſciſs, or equal to the product of the generating Circumference, multiplied by the fluxion of the Arch AM by ſuppoſition) will be equal to the Fluxion required.

Likewiſe, if the Figure AMP, be ſuppoſed to revolve about the line CB, drawn from a given point C perpendicular to AP, produced if neceſſary, as an axis; then the Product of the Surface deſcribed by PM, in that rotation, multiplied by the fluxion of the abſciſs CP, will be the Fluxion of the Solid generated by the figure AMP: and if the line MQ be drawn perpendicular to CB; the Product of the Surface, generated by the line QM about the axis CB, multiplied by the Fluxion of the abſciſs CQ, will be the Fluxion of the Solid generated by the outward ſpace CAMQ: and laſtly, the Product of the circumference deſcribed by the point M, in that rotation, multiplied by the fluxion of the arch AM, will be the Fluxion of the Surface of thoſe Solids.

If the right line AM continually terminated by the Curve AM, be imagined to revolve about the vertex A as a center, with an uniform motion; then the Fluxion of the Area AMA, generated by AM in that motion, will be equal to the Product of the generating line AM, multiplied by half the circular velocity of the point M. For the velocity of each particle of the line AM, is always* as its diſtance from the center A; and therefore the Velocities of all the particles will be to each other, as the Terms of an arithmetical Progreſſion: Conſequently the mean Velocity (which is equal to half the Velocity of the point M) being multiplied by the generating line, will be equal to the Sum of all the Products of the particles, each of them multiplied by its Velocity, that is, equal to the Fluxion of the ſpace AMA. *W.W.D.*

* *Art.* 188.

COROLLARY I.

230. IF AP $=$ EM $= x$, PM $=$ CQ $= y$, CP $= u$; then will $y\dot{x}$ be the Fluxion of the Area AMP; and by the ſame reaſon $x\dot{y}$, will be the Fluxion of the Area AEM: and becauſe the line AC or QE is conſtant, the Fluxion of the rectangle AQ will be $=$ QE $\times \dot{y} = u\dot{y} - x\dot{y}$; and therefore the Fluxion of the ſpace CAMQ will be equal to $u\dot{y} - x\dot{y} + x\dot{y} = u\dot{y}$.

COR.

COROLLARY II.

231. IF the part MN of the Tangent at M, expreſſes the Fluxion of the arch AM, and if the lines Ns, Nm, are drawn perpendicular to AM and PM produced; then Ns will be the Fluxion or circular Velocity of the point M, with reſpect to the radius AM, and Mm, mN, the Fluxions of the * ſemi-ordinate PM and of the abſciſs AP. Now if the line Nm interſects the line AM in n; the ſimilar triangles MPA, Mmn, Nns, will give 1.° PM : PA :: Mm : mn $= \frac{x\dot{y}}{y}$, or Nm—nm $= \dot{x} - \frac{x\dot{y}}{y}$; 2°. AM : PM :: Nn : Ns, or AM×Ns (=PM ×Nn) $= y\dot{x} - x\dot{y}$. Therefore, $\frac{y\dot{x}-x\dot{y}}{2}$ will be the Fluxion of the Segment AMA.

* Art. 164, 165.

It is evident, that the difference between the Fluxions $y\dot{x}$, $\frac{\dot{y}x+y\dot{x}}{2}$, of the Space AMP, and of the triangle AMP, will be equal to the ſame Fluxion.

COROLLARY III.

232. IF from the center C of an Ellipſis or Hyperbola, the line CM be drawn at pleaſure, and from its interſection M with the Curve, the ſemi-ordinate MQ to the ſecond axis CB: then if CP=QM$=u$, PM = CQ$=y$; the Fluxion of the ſpace CAMQ will be $\dot{y}u$, and that of the triangle CQM $\left(\frac{uy}{2}\right)$ will be $\frac{u\dot{y}+y\dot{u}}{2}$; and therefore their difference $\frac{\dot{y}u-y\dot{u}}{2}$ will be the Fluxion of the Sector CAM. FIG. 153, 154.

COROLLARY IV.

FIG. 150.

233. IF $\frac{c}{r}$ expreſſes the conſtant ratio of the Radius to its Circumference; then will $\frac{cy}{r}$ be the circumference whoſe radius is PM, and $\frac{cyy}{2r}$ the Area of its circle. Therefore $\frac{cyy\dot{x}}{2r}$, will be the Fluxion of the Solid, generated by the rotation of the Space AMP about the Axis AP; and by the ſame reaſon $\frac{cuu\dot{y}}{2r}$, will be the Fluxion of the Solid, generated by the rotation of the outward Space CAMQ about the axis CB.

COROLLARY. V.

234. BEcauſe $\frac{cuy}{r}$ expreſſes the Cylindric Surface, deſcribed by the rotation of the line PM about the axis CB; $\frac{cyu\dot{u}}{r}$, will be the Fluxion of the Solid generated by the rotation of the ſpace AMP, about that axis.

COROLLARY VI.

235. IF the Fluxion of the arch AM be $\dot{v}$; then will $\frac{cy\dot{v}}{r}$ be the Fluxion of the Surface, generated by the rotation of the arch AM about the axis AP, or which amounts to the ſame thing, of the Surface generated by the motion of the circumference whoſe radius is PM, along the arch AM: and $\frac{cu\dot{v}}{r}$ the Fluxion of the Surface, deſcribed by the rotation of that arch about the Axis CB.

Co-

If by the help of the Equation of a Curve, you find ſuch Expreſſions, as are equivalent to theſe general ones, and which contain but one ſort of variable quantities; then having found their fluents, by the Rules given in the firſt Section of this Book, you will have the Expreſſions required of the Areas, Surfaces and Solids, generated by that Curvilinear figure.

SCHOLIUM.

Though we have ſuppoſed in the preceding Problem and in all its Corollaries, that the ſemi-ordinates PM, QM, *fall at right angles to their abſciſſes* AP, CQ, *yet the fluxionary* Expreſſions *of the Areas, Surfaces and Solids, will be ſtill the ſame, when they contain any other given Angle, provided you ſubſtitute reſpectively into their fluents, inſtead of the abſciſs* AP, *or* CQ, *the line drawn from the beginning of that Abſciſs, perpendicular to the ſemi-ordinate* PM, *or* QM; *or inſtead of the ſemi-ordinates, the line drawn from the point* M, *perpendicular to the Abſciſs* AP, *or* CQ: *and therefore we ſhall hereafter uſe indifferently thoſe general Expreſſions in both Caſes, always reſerving in mind this Diſtinction.*

EXAMPLE I.

236. LET $x = y^m$ *be the Equation of the Curve* AM, *and let* Fig. 151. *it be required to find the* Expreſſion *of the Area* AMP.

If the fluxion $\dot{x} = m\dot{y}y^{m-1}$ of the Equation $x = y^m$, be multiplied by y, we ſhall have* $(y\dot{x} =)\ m\dot{y}y^m$, for the Fluxion of * Art. 230. that Area. Therefore the fluent* $\frac{m}{1+m}y^{m+1} = \frac{m}{1+m}xy$, will * Art. 213. be the Expreſſion required.

If the Fluxion $\dot{x} = m\dot{y}y^{m-1}$ be multiplied by $\frac{cyy}{2r}$, we ſhall have* * Art. 233. $\left(\frac{cyy\dot{x}}{2r} =\right) \frac{mc}{2r}\dot{y}y^{m+1}$, for the Fluxion of the Solid generated by the rotation of the ſpace AMP about the diameter AP; and* * Art. 213. $\frac{m}{2+m} \times \frac{c}{2r}y^{m+2} = \frac{m}{2+m} \times \frac{cx}{2r}yy$, for the Expreſſion of that Solid.

And if both Sides of the product $x\dot{x}=m\dot{y}y^{2m-1}$ of the Equations $x=y^m$, $\dot{x}=m\dot{y}y^{m-1}$, are multiplied by $\frac{cy}{r}$, we ſhall have
Art. 234. $\frac{c}{r}m\dot{y}y^{2m}\left(=\frac{c}{r}yx\dot{x}\right)$ for the Fluxion of the Solid, generated by the rotation of that Space about the tangent AT: and there-
Art. 213. fore,* $\frac{m}{1+2m}\times\frac{c}{r}y^{2m+1}=\frac{m}{1+2m}\times\frac{cxxy}{r}$, will be the Expreſſion of that Solid.

COROLLARY.

Fig. 152. 237. If m be negative; the Equation $x=y^{-m}$, will be that of all kinds of Hyperbolas with reſpect to the aſymptotes AT, AR; and the Expreſſions $\frac{-m}{1-m}xy$, $\frac{-m}{2-m}\times\frac{c}{2r}xyy$, $\frac{m}{1-2m}\times\frac{c}{r}xxy$, will become $\frac{m}{m-1}xy$, $\frac{m}{m-2}\times\frac{c}{2r}xyy$, and $\frac{m}{2m-1}\times\frac{c}{r}xxy$. Therefore, when $m>1$, then will, $\frac{m}{m-1}xy$ be the Expreſſion of the ſpace ARSMP, infinitely produced towards RS: and when $m<1$ $\frac{m}{m-1}$, xy will be the Expreſſion of the ſpace PMVT, infinitely produced towards VT; ſince that Expreſſion being negative, the ſpace ought to be taken ſo too; but when $m=1$, the Equation $x=y^{-1}$, will be that of the common Hyperbola, and $\left(\frac{m}{m-1}xy=\right)\frac{1}{0}xy$, ſhews, that the ratio of the circumſcribing parallelogram to that ſpace, is infinite.

COROLLARY II.

238. When $m>2$; then will $\frac{m}{m-2}\times\frac{c}{2r}xyy$ be the expreſſion of the Solid generated by the ſpace ARSMP; and when $m<2$, that of the Solid generated by the ſpace PMVT. But

if $m = 2$, then $\frac{m}{m-2} \times \frac{c}{2r} xyy$ will become $= \frac{c}{0r} xyy$. Which ſhews, that the ratio of the circumſcribing Cylinder to that Solid is infinite.

EXAMPLE II.

239. LET *the ſemi-ordinates* PM *fall at right-angles to* AP, *and let it be required to find the* Expreſſion *of the Arch* AM. Fig. 151.

The ſquare of the Fluxion of the Equation $x = y^m$, will be $\dot{x}^2 = m^2\dot{y}^2 y^{2m-2}$, and $\dot{x}^2 + \dot{y}^2 = \dot{y}^2 + m^2\dot{y}^2 y^{2m-2}$. Therefore $(\sqrt{\dot{x}^2 + \dot{y}^2} =) \dot{y}\sqrt{1 + m^2 y^{2m-2}}$, will be* the Fluxion of that Arch. Now if this Fluxion be compared with the general Form* $d\dot{z}z^r \times \overline{e + fz^n}^{\mathrm{m}}$, we ſhall have $r = 0$, $\mathrm{m} = \frac{1}{2}$, $n = 2m - 2$, and $\left(\frac{r+1}{n} =\right) s = \frac{1}{2m-2}$, or $m = \frac{2s+1}{2s}$: Likewiſe $\left(\frac{r+1+n\mathrm{m}}{n} =\right) t = \frac{m}{2m-2}$, or $m = \frac{2t}{2t-1}$; or if we ſuppoſe $-t$, $m = \frac{2t}{2t+1}$. And becauſe when s is a whole poſitive, or t a whole negative Number, the fluent of $\dot{y}\sqrt{1 + m^2 y^{2m-2}}$ may be found* in s or t Terms; therefore, if we ſubſtitute ſucceſſively for s, the Numbers 1, 2, 3, 4, &c. into the Equation $m = \frac{2s+1}{2s}$, and if the Exponent m expreſſes any of the Values found by that ſubſtitution, the Arch AM may be rectified.

* Art. 179.

* Art. 218.

* Art. 218.

For Example, if $s = 1$; then will $m = \left(\frac{2s+1}{2s}\right) \frac{3}{2}$; and therefore if $x = y^{\frac{3}{2}}$ or $xx = y^3$ be the Equation, the Fluxion $\dot{y}\sqrt{1 + m^2 y^{2m-2}}$ will become $\dot{y}\sqrt{1 + \frac{9}{4}y}$, whoſe fluent will be* $\frac{27}{8} \times \overline{1 + \frac{9}{4}y}^{\frac{3}{2}}$. But when $y = 0$, the arch AM will be likewiſe $= 0$; and as there remains $+ \frac{27}{8}$ of the fluent, by ſuppoſing $y = 0$; we muſt ſubſtract that quantity $\frac{27}{8}$ from the ſame, in order to have $\frac{27}{8} \times \overline{1 + \frac{9}{4}y}^{\frac{3}{2}} - \frac{27}{8}$, for the true Expreſſion required of the arch AM.

* Art. 214.

N. B. If the Numbers 1, 2, 3, *&c.* are ſubſtituted ſucceſſively inſtead of t, into the Equation $m = \frac{2t}{2t+1}$, we ſhall find the ſame values for m as before, except that the Fraction will be reverſed.

And becauſe the Numerator $2t$ of the Fraction $\frac{2t}{2t+1}$ is leſs than its Denominator $2t+1$; the Curve AM, will be convex next to the Abſciſs AP in this Caſe; which I prove thus:

Suppoſe $m = \frac{r}{n}$; then will $x = y^{\frac{r}{n}}$ be the Equation of the Curve AM; and * $\left(\dot{x} = \frac{r}{n} y^{\frac{r}{n}} = \right) \frac{r}{n} x$ will be the Expreſſion of the ſub-tangent. Now if $r > s$, the ſub-tangent will be greater than the Abſciſs AP; and therefore, the Curve AM will be concave next to AP. But if $r < s$: the ſub-tangent will be leſs than AP, and conſequently the Curve AM will be convex next to AP.

EXAMPLE III.

240. TO *find the* Expreſſion *of the Surface deſcribed by the Rotation of the Arch* AM *about the Axis* AP.

It is evident, that * $\frac{c}{r} y\dot{y}\sqrt{1+m^2y^{2m-2}}$ will be its Fluxion, which being compared with the general Form $d\dot{z}z^r \times \overline{e+fz^n}^{m}$ gives $\left(\frac{r+1}{n} = \right) s = \frac{1}{m-1}$, or $m = \frac{s+1}{1}$, and $\left(\frac{r+1+nm}{n} = \right) -t = \frac{m+1}{2m-2}$, or $m = \frac{2t-1}{2t+1}$. Therefore, ſubſtituting ſucceſſively for s, the Numbers 1, 2, 3, 4, *&c.* into the Equation $m = \frac{s+1}{s}$, we ſhall find ſuch Values for the Exponent m, which being taken ſucceſſively for the ſame, the fluent of the Fluxion $\frac{c}{r} y\dot{y}\sqrt{1+m^2y_{2m-2}}$, will be found in s terms, and ſubſtituting ſucceſſively for t, the Numbers 1, 2, 3, 4, *&c.* into the

* *Art.* 180.

* *Art.* 235.

the Equation $m = \frac{2t-1}{2t+1}$, we ſhall find ſuch values for m, which being taken ſucceſſively for the ſame, the fluent of $\frac{c}{r}y\dot{y}\sqrt{\overline{1+m^2y^{2m-2}}}$ will be found in t terms.

N. B. Becauſe the numerator of the fraction $\frac{2t-1}{2t+1}$, is leſs than its denominator, the values found for m, by means of the Equation $m = \frac{2t-1}{2t+1}$, will give the Caſes, in which the Surface generated by the rotation of the arch AM about the axis, towards which it is convex, may be found in a finite number of terms.

For Example, if $s=1$, then will $m \left(=\frac{s+1}{s}\right) = 2$, and $x=y^2$ will be the Equation of the Parabola, the parameter of which being $=1$; and the Fluxion $\frac{c}{r}y\dot{y}\sqrt{\overline{1+m^2y^{2m-2}}}$ will become $= \frac{c}{r}y\dot{y}\sqrt{\overline{1+4yy}}$; whoſe fluent will be $\frac{c}{12r} \times \overline{1+4yy}^{\frac{3}{2}}$. But becauſe when $y=0$, that Surface is likewiſe $=0$; and ſince there remains $+\frac{c}{12r}$ of the fluent, by ſuppoſing $y=0$, we muſt ſubtract that quantity $\frac{c}{12r}$ from the ſame, in order to have $\frac{c}{12r} \times \overline{1+4yy}^{\frac{3}{2}} - \frac{c}{12r}$ for the true Expreſſion required.

EXAMPLE IV.

241. TO *find the* Expreſſion *of any Space as* BCMP *of the common Hyperbola, terminated by the lines* BC, PM, *parallel to the aſymptote* AR. FIG. 152.

Let AB$=$BC$=1$, BP$=x$, PM$=y$; then will $\frac{1}{1+x}=y$ be the Equation, and $(y\dot{x}=)\ \frac{\dot{x}}{1+x}$ the Fluxion of that Space, which being reduced into an infinite Series by a continual diviſion: and the fluent of each term * found, gives $x-\frac{1}{2}x+\frac{1}{3}x^3+\frac{1}{4}x^4+\frac{1}{5}x^5-$ *&c.* for the Expreſſion required.

* *Art.* 213.

When PM falls between the aſymptote AR and the line BC, that is, if Ap$=x$, and pm$=y$; then will $\frac{1}{1-x}=y$ be the Equation, and $x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+$ *&c.* be the Expreſſion.

Now half the ſum of theſe Series, will be $x+\frac{1}{3}x^3+\frac{1}{5}x^5+\frac{1}{7}x^7+\frac{1}{9}x+$ *&c.* and half their Difference $\frac{1}{2}x^2+\frac{1}{4}x^4+\frac{1}{6}x^6+\frac{1}{8}x^8+$ *&c.* and when $x=\frac{1}{10}=0.1$, the firſt will become $0.1+\frac{0.001}{3}+\frac{0.0^{4}1}{5}$ $\frac{0.0^{6}1}{7}$ &c. and the laſt * $\frac{0.01}{2}+\frac{0,0^{3}1}{4}+\frac{0.0^{5}1}{6}+$ &c. which being reduced into decimals will give the following numbers.

0.1000000000000	0.0050000000000
3333333333	250000000
20000000	1666666
142857	12500
1111	100
9	1
0.1003353477310	0.0050251679267

Theſe two Sums being added together will give C, 0,1053605156577 &c. for the Expreſſion of the Space BCmp; and their Difference will give D, 0,0953101798043 &c. for the Expreſſion of the Space BCMP.

---

* 0^3 Stands for 000, 0^5 ſtands for 00000 *&c.*

COROLLARY I.

242. BEcauſe, when the Sectors mAC, mAM, mAN &c. or the correſpondent Spaces mCBp, mMPp, mNQp, (which are * equal to thoſe Sectors) are in an arithmetical Progreſſion, the correſpondent abſciſſes Ap, AB, AP, AQ, &c. will be * in a geometrical Progreſſion; it follows, that the ſaid Spaces are the Logarithms of their correſpondent abſciſſes; and therefore C, is the Logarithm of $Ap = 1 - \frac{1}{10} = 0.9$; and D the Logarithm of $AP = 1 + \frac{1}{10} = \frac{11}{10}$.

* Art. 114.

* Art. 110.

COROLLARY II.

243. IF we ſay, as the Expreſſion D, of the Hyperbolical ſpace BCMP, is to the Rectangle ABCD $(=1)$ or power of the Hyperbola, ſo is the Logarithm † .0413926851582 of $AP = \frac{11}{10}$, (equal to the Difference of the Logarithm of 11 and that of 10) to a fourth number $r = 0.434294481903$ &c. $=$ ABCD, this fourth number is called the *Logarithmical Modulus*: and if we divide Unity by that number, the Quotient $\frac{1}{r} = R$ 2.302585092994 &c. is called the *reciprocal Modulus*.

COROLLARY III.

244. HENCE, the Rectangle AC, is to any hyperbolical Space BCNQ, as the *Modulus* r, is to the tabular Logarithm of AQ: and on the contrary, The *Modulus* r, is to the Tabular Logarithm of any abſciſs AQ, as the Rectangle AC, is to the Expreſſion of the hyperbolical Space BCNQ correſpondent to that abſciſs. Therefore the power of the hyperbola, that is, the rectangle AC being given, the Space BCNQ may be found when the Logarithm of AQ is given; or that Logarithm may be found, when the Space BCNQ is given. For Example let L expreſs the Tabular Logarithm of AQ; then if we make $r\left(=\frac{1}{R}\right) : L :: ABCD : \frac{1}{r}L \times ABCD = RL \times ABCD$; this

---

† See *Sherwin's* Mathematical Tables, Page 18.

fourth term will be the Expreſſion of any hyperbolical ſpace ſuch as BCNQ. *This muſt be well obſerved, becauſe it will ſerve hereafter as a* Canon, *for finding the Expreſſions of ſuch Spaces, or of any Spaces, which may be reduced to them.*

COROLLARY IV.

245. BEcauſe when AB is $=1$, the ſpace BCMP is the Logarithm of AP, and the ſpace BCNQ, that of AQ: it follows, that the difference PMNQ of thoſe ſpaces is the Logarithm of $\frac{AQ}{AP}$.

COROLLARY V.

246. SInce $\frac{\dot{x}}{1+x}$ is the Fluxion of the ſpace BCMP, that is, the Fluxion of the Logarithm of AP, it follows, that the Fluxion of the Logarithm of any Quantity (AP) is equal to the Fluxion ($\dot{x}$) of that quantity, divided by the ſame ($1+x$). *This muſt be well remembred, becauſe it will be of great Uſe hereafter.*

EXAMPLE V.

Fig. 153. 247. TO *find the* Expreſſion *of a Circular Arch* AM, *whoſe Radius* AC *and Tangent* AT *are given.*

Let AC be $=1$, AT $=z$, tT $=\dot{z}$, and draw tr perpendicular to the Secant CT: then will CT $=\sqrt{1+zz}$, and becauſe of the right angled ſimilar Triangles CAT, Trt, we ſhall have CT : CA :: Tt : tr $=\frac{\dot{z}}{\sqrt{1+zz}}$; and the Radius CT,
* Art. 188. will be to the Radius CM,* as the circular velocity tr of the point T, is to the circular velocity $\frac{\dot{z}}{1+zz}$ of the point M, which is equal to the fluxion of the arch AM. Now that fluxion being reduced into an infinite Series, and the fluent of each term found,

will give $z \times$ into $1 - \frac{1}{3}z^2 + \frac{1}{5}z^4 - \frac{1}{7}z^6 + \frac{1}{9}z^8 -$ &c. for the Expreſſion required.

It is evident, that when $AT = z = \sqrt{\frac{1}{3}}$; the arch AM will contain 30 Degrees. For the ſine PM of an angle of 30 Degrees is $= \frac{1}{2}$, and the co-ſine $CP = \frac{1}{2}\sqrt{3}$; therefore $CP : PM :: CA : AT = \frac{1}{\sqrt{3}} = \sqrt{\frac{1}{3}}$. Conſequently ſubſtituting for z and its powers their values, we ſhall have $\sqrt{\frac{1}{3}} \times$ into $1 - \frac{1}{3.3} + \frac{1}{5.9} - \frac{1}{7.27} + \frac{1}{9.81} -$ &c.

If this Series be multiplied by 6, we ſhall have $(6\sqrt{\frac{1}{3}} =)$ $\sqrt{12} \times$ into $1 - \frac{1}{3.3} + \frac{1}{5.9} - \frac{1}{7.27} + \frac{1}{9.81} -$ &c. for the Expreſſion of the ſemi-circumference. Or if $A = \sqrt{12}$, $B = \frac{1}{3}A$, $C = \frac{1}{3}B$, $D = \frac{1}{3}C$, $E = \frac{1}{3}D$, *&c.* this Series will become $A - \frac{1}{3}B + \frac{1}{5}C - \frac{1}{7}D + \frac{1}{9}E -$ *&c.* this laſt Series being reduced into decimals will give,

3,464101615137775	,384900179459759
76980035891950	18328579974270
4751854067400	1295960200200
365527235950	105596757050
31057869720	9262873420
2793565000	850215430
260732730	80473060
24974390	7787710
2438570	766410
241660	76420
24230	7700
2450	780
250	80
20	
+3,546233172182070	—,404640518592280
—,404640518592280	
3,14159265358979. Difference or value of the Semicircumference.	

Therefore the Radius is to the Semi-circumference, or the Diameter is to the whole Circumference, as Unity is to the number 3.1415926 *&c.*

COR-

COROLLARY I.

248. HEnce the number 3,141592 &c. is to unity, as 180 degrees is to the number and parts of degrees contained in an arch equal to the Radius; which will be $=k$. 57,2957795130 &c.

This laſt number k, is called the *Trigonometrical Modulus:* and if we divide unity by this number, or 3,141592 &c. by 180, the Quotient ,01745329251994$3=K=\frac{1}{k}$ is called the *Reciprocal Modulus.*

COROLLARY II.

249. HEnce may be found the Expreſſion of an arch, whoſe radius and ſine or co-ſine are given; and on the contrary, there may be found the Expreſſion in degrees of an arch, whoſe radius and Expreſſion in numbers are given. For example, let D expreſs any arch as AM in degrees, and let $k=57,295779$ &c. then if we ſay, as the Radius AC expreſſed in degrees $=k$ is to the arch $AM=D$ expreſſed in degrees, ſo is the Radius AC ($=a$) expreſſed in numbers, to the arch $AM=\frac{a}{k}D$ expreſſed in numbers, or becauſe * $\frac{1}{k}=K$; we ſhall have $\frac{a}{k}D=aDK$. * *Art.* 248.

EXAMPLE VI.

FIG. 153. 154. 250. TO *find the* Expreſſion *of the Elliptical or Hyperbolical Sector* CAM.

* *Art.* 3. The fluxion of the Equation * $\pm\frac{aa}{bb}yy=aa-uu$, will be $\pm\frac{aa}{bb}y\dot{y}=-u\dot{u}$, or $\pm\frac{aa}{bbu}y\dot{y}=-\dot{u}$, ſubſtituting for $\dot{u}$ its

value into the fluxion * $\frac{\dot{y}u-y\dot{u}}{2}$ of that Sector, it will become * Art. 232. $=\frac{\dot{y}}{2u}\times\overline{uu\pm\frac{aa}{bb}yy}$, or becauſe $uu\pm\frac{aa}{bb}yy=aa$, and $u=\frac{a}{b}\sqrt{bb\mp yy}$; it will be $=\frac{aa}{2u}\dot{y}=\frac{ab\dot{y}}{2\sqrt{bb\mp yy}}$.

Now becauſe $\frac{bb\dot{y}}{2\sqrt{bb-yy}}$ is the Fluxion of a circular Sector, if we divide this Fluxion by $\frac{b}{2}$, we ſhall have $\frac{b\dot{y}}{\sqrt{bb-yy}}$ for the Fluxion of the circular arch, whoſe Radius and ſine are as b to y. Therefore if D expreſſes the arch in degrees, then will * bDK, * Art. 249. be the fluent of $\frac{b\dot{y}}{\sqrt{bb-yy}}$ and $\frac{ab}{2}$ DK, the expreſſion required of the Elliptical Sector CAM.

And if L expreſſes the Logarithm of $\frac{y+\sqrt{bb+yy}}{b}$; then will * $\frac{ab}{2}$ LR be the fluent of $\frac{ab\dot{y}}{2\sqrt{bb+yy}}$, or the Expreſſion of the * Art. 244. Hyperbolical Sector CAM.

For the Fluxion of the Log. of $\frac{y+\sqrt{bb+yy}}{b}$, will be * equal * Art. 246. to the Fluxion $\frac{\dot{y}\sqrt{bb+yy}+\dot{y}y}{\sqrt{bb+yy}}$ of that quantity divided by the ſame $y+\sqrt{bb+yy}$, that is, $=\frac{\dot{y}}{\sqrt{bb+yy}}$; and therefore $\frac{ab}{2}\times\frac{\dot{y}}{\sqrt{bb+yy}}$ will be the Fluxion propoſed.

If $y=b\sqrt{\frac{1}{4}}$, then if we ſay b, is to $b\sqrt{\frac{1}{4}}$, as the Radius in the Tables of ſines and tangents, is to the ſine of the arch whoſe ſine and radius are as y to b; or if we take the Logarithms of theſe numbers; then becauſe $b : b\sqrt{\frac{1}{4}} :: 1 : \sqrt{\frac{1}{4}}$; and Log. $\sqrt{\frac{1}{4}}=-624693$, and the Log. of the Radius is 10.0000000; the difference of theſe Logarithms $=9.9375307$, will be the Log. of the ſine of the angle ſought; which will be found to be $=60$ degrees:

* *Art.* 248. and becauſe K $=$* 0.0174532925, and D$=60$; we ſhall have $\frac{1}{2}ab$DK$=ab\times$ into 0.523598775 &c, for the Expreſſion of the elliptical Sector CAM, in this Caſe.

FIG. 254. Likewiſe, the Log. of $\frac{y+\sqrt{bb+yy}}{b}=\sqrt{\frac{3}{4}}+\sqrt{\frac{7}{4}}=2.188$ is $=$

* *Art.* 243. 0.3400473, and becauſe R $=$ * 2.302585092994; we ſhall have $\frac{1}{2}ab$RL $=ab\times$ into 0.3914939219 &c. for the Expreſſion of the hyperbolical Sector CAM in this Caſe.

EXAMPLE VII.

FIG. 153, & 154. 251. TO *find the* Expreſſion *of the ſame Sector, by taking the Tangent* AT (z) *for the variable quantity.*

Let the line CM (produced in the Ellipſis) meet the tangent in T; then the ſimilar triangles CPM, CAT, will give $y=\frac{uz}{a}$; whoſe fluxion is $\dot{y}=\frac{u\dot{z}+\dot{u}z}{a}$. Therefore ſubſtituting for y, $\dot{y}$,

* *Art.* 232. their values into the fluxion * $\frac{\dot{y}u-y\dot{u}}{2}$, it will become $=$ $\frac{uu\dot{z}}{2a}$, or becauſe $\pm\frac{aa}{bb}yy=aa-uu$, equal to $\pm\frac{uu}{bb}zz$ (ſince $y=\frac{uz}{a}$); we ſhall have $uu=\frac{aabb}{bb\pm zz}$. And conſequently $\left(\frac{uu\dot{z}}{2a}=\right)\frac{abb\dot{z}}{2bb\pm 2zz}$ will be the fluxion of that Sector.

* *Art.* 247. Now if D expreſſes in degrees, the arch whoſe fluxion is *

* *Art.* 249 $\frac{bb\dot{z}}{bb+zz}$, and whoſe radius and tangent are as b to z, then will * bDK be the fluent of $\frac{bb\dot{z}}{bb+zz}$: and therefore $\frac{1}{2}ab$DK will be the Expreſſion of the Elliptical Sector CAM required.

* *Art.* 244. And if L expreſſes the Log. of $\frac{b+z}{\sqrt{bb-zz}}$; then will * $\frac{1}{2}ab$RL be the Expreſſion of the Hyperbolical Sector CAM. For the

* *Art.* 246. Fluxion of the Log. of $b+z$, will be* $\frac{\dot{z}}{b+z}=\frac{b\dot{z}-z\dot{z}}{bb-zz}$, and that of

the Log. of $\sqrt{bb-zz}$ will be $=\frac{-z\dot{z}}{bb-zz}$; and therefore their difference will be $=\frac{b\dot{z}}{bb-zz}$.

COROLLARY I.

252. HEnce, if in the Hyperbola, the Expreſſion of the Sector CAM be added to the triangle CQM, or* ſubſtracted from the triangle CPM; we ſhall have the Expreſſion of the Space CAMQ, or of the Space AMP. * *Art.* 249.

And in the Ellipſis, if the triangle CQM be added to, or the triangle CPM ſubtracted from the Expreſſion of the Sector CAM; we ſhall have that of the Space CAMQ, or of the Space AMP. FIG. 153.

If $z=\frac{1}{2}b$; then if in the Ellipſis AC (b), be to AT ($\frac{1}{2}b$), as the radius in the tables of ſines (1000000), is to a fourth number 500000, which is equal to the tangent of an angle of 26 degrees 34 minutes $=26,56666.=D$; and becauſe K =* 0.01745329 2519943; we ſhall have $\frac{1}{2}abDK=ab\times$ into 0.2318379 &c. for the Expreſſion of the Elliptical Sector CAM. * *Art.* 248.

And in the Hyperbola, we ſhall have $\frac{b+z}{\sqrt{bb-zz}}=\sqrt{3}$: and becauſe 0.301030 is the Log. of 3, half that number 0.150515, will be the Log. of $\sqrt{3}$, that is, $=L$; and becauſe R=* 2.302585092994; $\frac{ab}{2}LR$ will be $=ab\times$ into 0.17328679 &c. equal to the Expreſſion of the Hyperbolical Sector CAM. * *Art.* 243.

EXAMPLE. VIII.

253. L*ET the Curve* AME *be the* Ciſſoid *whoſe Property is ſuch, that any of its Semi-ordinates as* PM *to the axis* AB, *is always a third proportional, to the ſemi-ordinate* PN *of its generating Circle* ANB, *and the abſciſs* AP; *it is required to find the* Expreſſion *of the Space* AMP. FIG. 155.

If AB$=a$, AP$=x$, PM$=y$; then will PN$=\sqrt{ax-xx}$, and $y=\frac{xx}{\sqrt{ax-xx}}$ will be the Equation of the Ciſſoid. Therefore $(y\dot{x}=)\ \frac{xx\dot{x}}{\sqrt{ax-xx}}$, will be the fluxion of the Space AMP. Now becauſe $\dot{x}\sqrt{ax-xx}$ is the Fluxion of the circular Space ANP; $3\text{ANP}-2x\sqrt{ax-xx}$, will be the Expreſſion required. For the fluxion of this Fluent is the very ſame as that propoſed.

EXAMPLE IX.

254. *TO find the* Expreſſion *of the Solid generated by the rotation of the ſpace* AMP, *about the line* AB *as an axis.*

* *Art.* 233. If $2r=a$; then will * $\left(\frac{cyy\dot{x}}{2r}=\right)\frac{c\dot{x}x^3}{aa-ax}=\frac{aac\dot{x}}{a-x}-\frac{cxx\dot{x}}{a}$ $-cx\dot{x}-ac\dot{x}$, be the Fluxion of the Solid. Therefore if L$=$
* *Art.* 244. Log. $\frac{a-x}{a}$; we ſhall have * $aac\text{LR}-\frac{cx^3}{3a}-\frac{cxx}{2}-acx$, for the Expreſſion required; and when $x=\frac{1}{2}a$, then will L$=$
* *Art.* 243. Log. $\frac{1}{2}=0.30103$, and becauſe R is $=$*2.30258 &c. aacLR will be $=aac$ × into 0.69314719 &c., and $aac\text{LR}-\frac{2aac}{3}$, will then be the Expreſſion required.

EXAMPLE. X.

FIG. 137. 255. *TO find the* Expreſſion *of the Space* FPMA *of the Logarithmical Curve, infinitely produced towards* FA.

* *Art.* 201. Becauſe* $a\dot{y}=y\dot{x}$, we ſhall have $ay=$TM×PM, for the Expreſſion required. If another Perpendicular QN be drawn to the Aſymptote FP; then will PT×$\overline{\text{QN}-\text{PM}}$ be the Expreſſion of the Space NQPM.

If $r = a =$ TP; then will * $\left(\frac{cyy\dot{x}}{2r} = \right)\frac{cy\dot{y}}{2}$ be the Fluxion of * *Art.* 233. the Solid generated by the rotation of the Space FPMA about the aſymptote FP, and $\frac{c}{4}yy = \frac{c}{4}\overline{\text{PM}}^2$ its Expreſſion: and therefore $\frac{c}{4} \times \overline{\text{QN}}^2 - \frac{c}{4}\overline{\text{PM}}^2$, that of the Solid generated by the Space NQPM in that rotation.

EXAMPLE XI.

256. T*O find the* Expreſſion *of the Space* PFM *of the Spiral,* Fig. 139. *whoſe Equation is* * $ma\dot{y}y^m = \dot{x}$. * *Art.* 203.

It is evident that the product of the radius PM multiplied by half the circular velocity ($\dot{x}$) of the point M*, will be the Fluxion of that Space. * *Art.* 229. Therefore that Fluxion, will be $\frac{1}{2}ma\dot{y}y^{m+1}$ $\left(= \frac{y\dot{x}}{2}\right)$, and $\frac{ma}{2m+4}y^{m+2}$ the Expreſſion required.

When $m = -1$; then will $a\dot{y} = y\dot{x}$ be the Equation, Fig. 140. which is that of the reciprocal Spiral *, and $\frac{ay}{2} = \frac{1}{2}\text{AB} \times \text{PM}$ * *Art.* 204. the Expreſſion.

If $a\dot{y} = \dot{x}$ be the Equation, which is that of the Logarithmical Spiral * PFM, which turns an infinite number of times about Fig. 138. the pole P before it arrives at M; then will $\frac{ayy}{4} = \frac{1}{4}\text{TP} \times \text{PM}$ * *Art.* 202. be the Expreſſion required.

EXAMPLE XII.

257. L*ET the right Cylinder* LAa, *be cut by an oblique Plane* BDNb, *paſſing thro' the center* C *of the circular baſe* Fig. 156. BAba; *it is required to find the* Expreſſion *of the part* BADNb *call'd the* Ungula, *or* Hoof.

It is evident, that the interſections of the Solid and of all the planes as CAD, PMN, perpendicular to the baſe of the Cylinder, and whoſe common Sections CA, PM with the baſe, are perpendicular to Bb, will be right-angled ſimilar triangles CAD, PMN. If therefore $CA=CB=a$, $bP=x$, $PM=y$, and the altitude of that Solid $\overline{AD=b}$; then by the property of the Circle, we ſhall have $y=\sqrt{2ax-xx}$, and by the ſimilarity of triangles CA : AD : : PM : MN $=\frac{by}{a}=\frac{b}{a}\sqrt{2ax-xx}$; and ſo $\frac{\dot{x}}{2}$ PM × MN $=bx\dot{x}-\frac{bxx\dot{x}}{2a}$, will be the Fluxion of the Part bNMP, and the fluent $\frac{bxx}{2}-\frac{bx^3}{6a}$ its Expreſſion: and when $x=2a$, we ſhall have $\frac{2aab}{3}$ for the Expreſſion of the whole Solid BADNb.

EXAMPLE XIII.

258. LET *it be required to find the* Expreſſion *of the convex Surface of that Solid.*

It is evident, that the Product $b\dot{x}$ $(=\frac{b\dot{x}}{y}\sqrt{2ax-xx})$ of MN multiplied by the Fluxion $\frac{a\dot{x}}{y}$ of the circular arch bM will be the fluxion of the part bNMb; and therefore, the fluent bx, will be its Expreſſion, and $2ab=2CB\times AD$, that of the whole Surface.

EXAMPLE XIV.

Fig. 157

259. TO *find the* Expreſſion *of the Solid generated by the rotation of the Semi-Ellipſis* ABa *about the Tangent* AT.

Draw any where a line Lm parallel to the diameter Aa, and from its interſections M, m, with the Curve, the Semi-ordinates PM, pm to Aa; then the ſum of the Surfaces deſcribed by the ſemi-ordinates PM, pm, in that rotation, multiplied by the Fluxion of the abſciſs AP will be the Fluxion of any two parts of the

Solid generated by the Spaces AMP, amp. If therefore $r=a$; then becauſe $AP+Ap=2a$; $2c$ will be the Sum of the circumferences deſcribed by the radii AP, Ap; and * $(2cy\dot{x}=)\ \frac{2cb\dot{x}}{a}\times$ * Art. 234.

$\sqrt{2ax-xx}$, will be the fluxion of thoſe parts, ſince $y=\frac{b}{a}\sqrt{2ax-xx}$.

But becauſe $(y\dot{x})\ \frac{b\dot{x}}{a}\sqrt{2ax-xx}$ is the fluxion of the Space AMP;

$2c$ AMP will be the fluent of that fluxion, and $2c$ × into the area of the elliptical quadrant ABC, will be the Expreſſion required of the whole Solid.

EXAMPLE XV.

260. L*ET it be required to find the* Expreſſion *of the convex Surface of that Solid.*

The product of the Sum of the Circumferences, whoſe radii are AP, Ap, multiplied by the fluxion ($\dot{z}$) of the elliptical arch AM, will give $2c\dot{z}$, for the Fluxion of any two parts of that Surface deſcribed by the arches Am, ma: the fluent of which is $2cz$; and $2c$ × into the Curve AB of the quadrant, will be the Expreſſion required.

EXAMPLE XVI.

261. T*O find the* Expreſſion *of the arch of the common Parabola,* Fig. 151.
whoſe equation is * $2px=yy$. * Art. 1.

Let AP be the axis; then the Fluxion $p\dot{x}=y\dot{y}$, will give $\dot{x}^2=\frac{y^2\dot{y}^2}{pp}$; and $\sqrt{\dot{x}^2+\dot{y}^2}=\frac{\dot{y}}{p}\sqrt{pp+yy}$ for the Fluxion of the arch.

Therefore, if $L=\text{Log.}\ \frac{y+\sqrt{pp+yy}}{p}$, then will * $\frac{1}{2}pLR$ be the * Art. 244.

fluent of $\frac{1}{2}p\dot{y}\times\overline{pp+yy}^{-\frac{1}{2}}$ and $\frac{1}{2p}y\sqrt{pp+yy}+\frac{1}{2}pLR$ will be the Expreſſion required.

When $y=2p$, then will $L=\text{Log. } 2+\sqrt{5}=4,236$ be $=0.6269560$ and becauſe $R = 2.302585$ *&c.* we ſhall have $p\sqrt{5}+p\times$ into $\cdot 0.72180974$ *&c.* for the Expreſſion required of the arch AM.

EXAMPLE XVII.

FIG. 153. 262. TO *find the* Expreſſion *of the Elliptical arch* BM, *terminated by the ſecond axis* BC *and any ordinate* PM *to the firſt* AC.

* *Art.* 3 If we ſuppoſe AC$=1$; then will * $yy=bb\times\overline{1-uu}$, or $y=b\sqrt{1-uu}$ be the Equation of the Ellipſis, whoſe Fluxion will be $\dot{y}=\frac{-bu\dot{u}}{\sqrt{1-uu}}$. And therefore $\dot{u}^2+\dot{y}^2=\dot{u}^2+\frac{bbuu\dot{u}^2}{1-uu}=\dot{u}^2\times\frac{1-uu+bbuu}{1-uu}$, or if we ſuppoſe $1-bb=dd$; then $\sqrt{\dot{u}^2+\dot{y}^2}=\dot{u}\sqrt{\frac{1-dduu}{1-uu}}$ will be the Fluxion of that arch. Now becauſe

* *Art.* 222. $\overline{1-dduu}^{\frac{1}{2}}=$ * $1-\frac{dd}{2}uu-\frac{1.1}{2.4}d^4u^4-\frac{1.1.3}{2.4.6}d^6u^6-$ *&c.*, and $\overline{1-uu}^{-\frac{1}{2}}=1+\frac{1}{2}uu+\frac{1.3}{2.4}u^4+\frac{1.3.5}{2.4.6}u^6$ *&c.*, the product of theſe two Series being multiplied by $\dot{u}$, and the fluent of each term found, will give for the Expreſſion deſired.

$$
\begin{array}{llll}
u-\frac{1.1}{2.3}ddu^3 & -\frac{1.1.1}{2.4.5}d^4u^5 & -\frac{1.1.3}{2.4.6.7}d^6u^7 & -\text{\&c} \\
\quad+\frac{1.1}{2.3}u^3 & -\frac{1.1}{2.2.5}d^2u^5 & -\frac{1.1.1}{2.2.4.7}d^4u^7 & -\text{\&c.} \\
 & +\frac{1.3}{2.4.5}u^5 & -\frac{1.1.3}{2.2.4.7}d^2u^7 & -\text{\&c.} \\
 & & +\frac{1.3.5}{2.4.6.7}u^7 & -\text{\&c.} \\
 & & & +\text{\&c.}
\end{array}
$$

EX-

EXAMPLE XVIII.

263. TO *find the* Expreſſion *of the Hyperbolical arch* AM. FIG. 154.

Becauſe * $yy = bb \times \overline{uu - 1}$ is the Equation of the Hyperbola; * Art. 3. we ſhall find in the ſame manner as in the Ellipſis, by ſuppoſing $1 + bb = dd$, that $\dot{u}\sqrt{\frac{dduu - 1}{uu - 1}}$, is the Fluxion of the arch AM:

and becauſe $\sqrt{\overline{dduu - 1}} =$ * $du - \frac{1}{2du} - \frac{1}{2.4d^3u^3} - \frac{3}{2.4.6d^5u^5}$ * Art. 222.

$- \frac{3.5}{2.4.6.8d^7u^7}$ — &c. and $\frac{1}{\sqrt{uu - 1}} = u + \frac{1}{2u} + \frac{3}{2.4u^3} + \frac{3.5.}{2.4.6.u^5}$ + &c. the product of theſe Series multiplied by $\dot{u}$, and the fluent of each term found will give

$$\frac{1}{3}du^3 - \frac{u}{2d} + \frac{1}{2.4.d^3u} + \frac{1}{2.4.6.d^5u^3} + \frac{3}{2.4.6.8.d^7u^5} + \text{\&c.}$$

$$+ \frac{du}{2} + \frac{1}{2.2.du} + \frac{1}{2.2.4.3.d^3u^3} + \frac{3}{2.2.4.6.5.d^5u^5} + \text{\&c.}$$

$$- \frac{3d}{2.4.u} + \frac{3}{2.2.4.3.du^3} + \frac{3}{2.2.4.4.5.d^3u^5} + \text{\&c.}$$

$$- \frac{3.5.d}{2.4.6.3u^3} + \frac{1.3.5}{2.2.4.6.5.du^5} + \text{\&c.}$$

$$- \frac{1.3.5.7d}{2.4.6.8.5.u^5} + \text{\&c.}$$

$$- \text{\&c}$$

EXAMPLE XIX.

264. TO *find the* Expreſſion *of the Surface deſcribed by the rotation of the Elliptical arch* BM *about the axis* A a. FIG. 158, 159.

If $r = b$; then becauſe $y = b\sqrt{\overline{1 - uu}}$, we ſhall have * * Art. 262.

$\frac{b\dot{u}}{y}\sqrt{\overline{1 - dduu}}$ for the Fluxion of the arch BM, and * $\left(\frac{cy\dot{v}}{r} =\right)$ * Art. 235.

Mm cu

$c\dot{u}\sqrt{1-dduu}$ for the Fluxion of the Surface. Now if in the ſemi-conjugate CB to CA, be taken CD=CA, and with the ſemi-diameter CD, and with the Center C, be deſcribed the ſemi-circumference ADa of a Circle, when Aa is the firſt axis, or an Equilateral Hyperbola NDn, when Aa is the ſecond; then if CQ$=du$, and the ſemi-ordinate QN be drawn, $\frac{c}{d}\times$ CQND will be the Expreſſion required.

* Art. 4. For when Aa is the firſt axis; then will * $1-bb=dd$ be poſitive; and therefore $\dot{u}d\sqrt{1-dduu}$ will be the Fluxion of the circular ſpace CQND: and when Aa is the ſecond, $1-bb=dd$ will be negative, that is, $bb-1=dd$; and conſequently $\dot{u}d\sqrt{1+dduu}$ will be the Fluxion of the Hyperbolical Space CQND.

When Aa is the firſt axis and $u=1=$AC; then will CQ$=d$,
* Art. 4. that is, the point Q will coincide with the * Focus.

If D be the Expreſſion in degrees of a circular arch, whoſe radius and ſine are as 1 to du; then will $\frac{cu}{2}\sqrt{1-dduu}+\frac{c}{2d}$KD be the fluent of $c\dot{u}\sqrt{1-dduu}$: and if L be the Logarithm of $du+\sqrt{1+dduu}$; then will $\frac{cu}{2}\sqrt{1+dduu}+\frac{c}{2d}$RL be the fluent of $c\dot{u}\sqrt{1+duu}$. For the Fluxion of the Logarithm of $du+\sqrt{1+dduu}$ will be equal to the Fluxion $\frac{d\dot{u}\sqrt{1+dduu}+ddu\dot{u}}{\sqrt{1+dduu}}$ of that quantity divided by the ſame, that is, $=\frac{d\dot{u}}{\sqrt{1+dduu}}$. Therefore this fluxion multiplied by $\frac{c}{2d}$, being added to the Fluxion of $\frac{cu}{2}\sqrt{1+dduu}$, will give $c\dot{u}\sqrt{1+dduu}$, which is the Fluxion propoſed.

EXAMPLE XX.

265. TO *find the* Expreſſion *of the Surface deſcribed by the rotation of the Hyperbolical arch* AM, *about one of the axes.* Fig. 160.

If $r=b$, then becauſe $y=b\sqrt{uu\mp1}$, we ſhall have * $\frac{b\dot{u}}{y}\sqrt{dduu\mp1}$ for the fluxion of the arch AM and* $\left(\frac{cy\dot{v}}{r}=\right)c\dot{u}\sqrt{dduu\mp1}$ for the Fluxion of the Surface. Now if in the Axis of Revolution, be taken $CQ=du$, and the ſemi-ordinate QN be drawn; then will $\frac{c}{d}$ANQ or $\frac{c}{d}$ CANQ, be the Expreſſion required: Since $d\dot{u}\sqrt{dduu\mp1}$ is the Fluxion of the Space ANQ when -1, or of the Space CANQ when $+1$.

* Art. 263.

* Art. 235.

When the arch AM revolves about the firſt axis, and $u=1$, the Surface will become equal to nothing: but becauſe $CQ=du$ then becomes, $CQ=d$, that is, the point Q coincides with the Focus F, and ſo the Space AMP is not $=0$; therefore we muſt take away the part AEF, in order to have $\frac{c}{d}$ EFQN for the true Expreſſion.

If L=Log. of $du+\sqrt{dduu\mp1}$; then will $\frac{cu}{2}\sqrt{dduu\mp1}\mp\frac{c}{2d}$ RL be the Expreſſion required.

EXAMPLE XXI.

266. TO *find the* Expreſſion *of the arch* AM *of the Logarithmical Curve infinitely produced towards* A. Fig. 137.

Becauſe * $\frac{a\dot{y}}{y}=\dot{x}$ we ſhall have $\sqrt{\dot{x}^2+\dot{y}^2}=\frac{\dot{y}}{y}\sqrt{aa+yy}$, for the Fluxion of the arch; and if $z=\sqrt{aa+yy}$, that Fluxion may be changed into this other $\frac{zz\dot{z}}{zz-aa}=\dot{z}+\frac{aa\dot{z}}{zz-aa}$. Therefore if

* Art. 201.

L

* *Art.* 244. L =Log. of $\frac{z-a}{\sqrt{zx-aa}}=\frac{\sqrt{aa+yy}-a}{y}$; then will $\sqrt{aa+yy}+{}^{*}aLR$ be the Fluent required. For the Fluxion of the Logarithm of $z-a$, is $\frac{\dot{z}}{z-a}=\frac{z\dot{z}+a\dot{z}}{zz-aa}$ and that of the Log. of $\sqrt{zz-aa}$ is $\frac{z\dot{z}}{zz-aa}$; which being ſubſtracted from the firſt, will give $\frac{a\dot{z}}{zz-aa}$, for the Fluxion of the Log. of $\frac{z-a}{\sqrt{zz-aa}}$; and therefore $\dot{z}+\frac{aa\dot{z}}{zz-aa}=\frac{zz\dot{z}}{zz-aa}=\frac{\dot{y}}{y}\sqrt{aa+yy}$.

EXAMPLE XXII.

FIG. 138. 267. *TO find the* Expreſſion *of the arch* PFM *of the Logarithmical Spiral.*

* *Art.* 202. Let PM * : TM : : 1 : *m*, or $TM=my$; then becauſe when PM expreſſes its own Fluxion, TM expreſſes the Fluxion of the Curve, $m\dot{y}$ will be the Fluxion of that arch, and $my=TM$ the Expreſſion required.

FIG. 140. * *Art.* 204. If $^{*}n\dot{y}=y\dot{x}$ be the Equation; then will $\sqrt{\dot{x}^2+\dot{y}^2}=\frac{\dot{y}}{y}\sqrt{nn+yy}$ be the Fluxion: whoſe fluent will be found in the ſame manner as that in Article 266.

SECT.

SECTION III.

The Uſe of Fluxions in finding the Centers of Gravity.

DEFINITIONS.

1. The *Center* of *Gravity* of a Figure, is a point by which that Figure being ſuſtained or ſuſpended, all its parts remain at reſt or in equilibrium.

2. Any line drawn thro' the Center of Gravity of a Figure, in an horizontal plane, is call'd the *Axis* of *Equilibrium*.

3. The Propenſity or Tendency of a Body towards the Earth, with reſpect to its diſtance from its Axis of Equilibrium, is call'd the *Momentum* of that Body.

COROLLARY.

268. IT is evident from *Defin. 2d*, that if G be the Center of Gravity of the Figure PQRS, and dG the Axis of Equilibrium, the vertical plane paſſing thro' the axis, will divide the *Momentums* of all the Particles of the Figure, into two equal parts. For if the Sum of the *Momentums* on one ſide of the plane were greater than the Sum of thoſe on the other, the Figure would be more inclinable to move one way than the other, and ſo the point G would not be its Center of Gravity, which is againſt the Suppoſition, therefore *&c.* Fig. 162.

POSTULATUMS.

I. That Gravity is every where the ſame at a ſmall diſtance from the ſame point of the Surface of the Earth.

1. That the right Lines or Directions, in which Gravity tends towards the Earth, may be taken as parallel to each other, in a ſmall horizontal diſtance.

N. B. 1. Becauſe the weights or quantities of Matter of the ſame denſity, are proportional to their Bulks or Magnitudes, and as we ſhall here only conſider Bodies of the ſame denſity; we ſhall make uſe of their Bulks or Magnitudes inſtead of their Weights.

2. Although Lines, Areas and Surfaces, have really no Weights, yet as the Conſideration of their Centers of Gravity is uſeful, we ſhall make uſe of their contents or meaſures in the ſame manner as in the Solids, as ſo many Weights proportional to them.

THEOREM I.

Fig. 161. 269. IF *ſeveral Bodies or Weights* P, Q, R, S, *&c., are fixed to an inflexible line* AE *void of Gravity, in* A, B, E, F, *&c., and if that line be ſuſpended or ſuſtained by a point* G, *ſuch that the ſum of the Products of the Weights on one ſide, each of them being multiplied by its reſpective diſtance from that point, is equal to the ſum of the like products on the other; I ſay the point* G *will be the common Center of Gravity of thoſe Bodies or Weights.*

The farther the Body P is fixed from the point G, the greater
* *Art.* 188. power it will have* to move about that point, if not prevented by ſome contrary action; and the heavier the Body is, the greater power it has likewiſe; therefore its *Momentum*, with regard to the point G, will be as $P\times AG$.

The ſame thing may be proved with regard to any other Body, therefore $P\times AG+Q\times BG=R\times EG+S\times FG$ by ſuppoſition; that is, the ſum of the *Momentums* on one ſide, is equal to the ſum of the *Momentums* on the other: conſequently the point G is the common Center of Gravity of thoſe Bodies.

THEOREM II.

Fig. 162. 270. IF *ſeveral Bodies* P, Q, R, S, *&c., are fixed together ſo as to be in different Planes: the* Sum *of all the Products of each Body multiplied by its reſpective diſtance from a plane given in poſition, will be equal to the Product of the Sum of all thoſe Bodies, multiplied by the diſtance of their common Center of Gravity* G *from that plane, if they are all placed on one ſide: Or if ſome are placed*

placed on one ſide, and ſome on the other, the Difference *of the like Products of thoſe on one ſide, to thoſe on the other, will be equal to the ſaid Product.*

Through the Center of gravity G, draw an horizontal plane, in which draw the line Gg or Gd, parallel to the interſection DT or DF of that plane and the plane dT or DF given in poſition; likewiſe from the points P, Q, R, S, &c. where the directions of the Bodies meet the horizontal plane, draw the lines PA, QB, RE, SF, parallel to DT, interſecting DF in A, B, E, F, &c. and dG, in a, b, e, f, &c. Now it is evident (by *Poſt.* 1.) that the gravity will be the ſame, whether thoſe Bodies are fixed all in the ſame plane, or ſome under and ſome above it: therefore, if we conſider the line Gg as an axis of Equilibrium, we ſhall have * $P\times aG+Q\times bG=R\times eG+S\times fG$, or becauſe $aG=Dg-DA$, $bG=Dg-DB$, $eG=DE-Dg$, $fG=DF-Dg$; that Equation will become $P\times\overline{Dg-DA}+Q\times\overline{Dg-DB}=R\times\overline{DE-Dg}+S\times\overline{DF-Dg}$, or $P\times DA+Q\times DB+R\times DE+S\times DF=Dg\times\overline{P+Q+R+S}$. And if we conſider Gd as an axis of equilibrium, then will $P\times Pa+R\times Re=Q\times Qb+S\times Sf+$ &c, or becauſe $Pa=PA+Gg$, $Re=RE+Gg$, $Qb=QB-Gg$, $Sf=SF-Gg$, the laſt Equation will become this $P\times\overline{PA+Gg}+R\times\overline{RE+Gg}=Q\times\overline{QB-Gg}+S\times\overline{SF-Gg}$, or $-P\times PA+Q\times QB-R\times RE+S\times SF=Gg\times\overline{P+Q+R+S}$ + &c. W. W. D.

* *Art.* 269.

From whence we deduce this general Rule for finding the Diſtance of the Center of gravity of any figure, from a plane given in poſition.

Multiply each particle of the figure, by its diſtance from the plane given in poſition, and divide the Sum of all the Products (if the plane falls without the figure) or the difference (if it falls within the ſame) between the Products on one ſide and thoſe on the other, by the content of the Figure; then the Quotient will be the Diſtance required.

EXAMPLE I.

Fig: 163. 271. *To find the* Diſtance *of the Center of gravity of a Triangle* ABC *from the line* TAT, *parallel to its baſe* BC.

Let AE (a) biſect the parallels MN, BC, in P, and E; then if AP $= x$, BC $= b$; we ſhall have MN $= \frac{bx}{a}$. Therefore $\frac{bx\dot{x}}{a}$ will be the Fluxion of the Weights, which being multiplied by the diſtance AP, will give $\frac{bxx\dot{x}}{a}$ for the Fluxion of the *Momentums*; whoſe fluent $\frac{bx^3}{3a}$ divided by the ſum $\frac{bxx}{2a}$ of the Weights, will give $\frac{2x}{3}$ for the expreſſion of the ſaid diſtance of the triangle AMN, and $\frac{2a}{3} = \frac{2}{3}$AE for that of the Triangle ABC. Becauſe the line AE biſects all the parallels MN, the Center of gravity will be in that line; and therefore if AG $= \frac{2}{3}$AE, the point G will be the Center required.

COROLLARY.

Fig. 164. 272. LET the quadrilateral Figure ABCD, be divided into two triangles, whoſe Centers of gravity let be E and F: then if the point G be taken in the line EF, ſuch that the content of the whole figure, be to that of any of the parts as ABD, as the line EF, is to the diſtance GF of the point G from the Center of gravity of the other part; the point G will be the Center of gravity of the Figure. For if we conſider the contents of the triangles ABD, BCD, as fixed in the points E, F, to the line EF; then will * ABD : BCD : : GF : GE; and by compoſition ABCD : ABD : : EF : GF.

* Art. 269.

EX-

EXAMPLE II.

273. TO *find the* Diſtance *of the Center of gravity of the ſemi-Parabola* BAE, *from the tangent* AT, *whoſe Equation is* $x=y^m$. Fig. 165.

The fluxion $\dot{x}=m\dot{y}y^{m-1}$ being multiplied by y, gives ($y\dot{x}=$) $m\dot{y}y^m$ for the fluxion of the Weights; and therefore ($yx\dot{x}=$) $m\dot{y}y^{2m}$ will be that of the *Momentums*; whoſe Fluent $\frac{m}{2m+1}y^{2m+1}$ divided by the ſum of the Weights $\frac{m}{m+1}y^{m+1}$, gives $\frac{m+1}{2m+1}y^m=\frac{m+1}{2m+1}x$ for the Expreſſion of the diſtance required.

It is evident, that if we conſider the Weights $y\dot{x}$ to be united in the middle of the ſemi-ordinate PM; then the *Momentum* of that weight with regard to the axis AE, will be ($\frac{1}{2}y \times y\dot{x}=$)$\frac{m}{2}\dot{y}y^{m+1}$; conſequently its fluent $\frac{m}{2m+4}y^{m+2}$ divided by the Weights $\frac{m}{m+1}y^{m+1}$, will give $\frac{m+1}{2m+4}y$ for the Expreſſion of the Diſtance required from the axis AE.

If $AP=\frac{m+1}{2m+1}AE$, and $PG=\frac{m+1}{2m+4}EB$; the point G will be the Center of gravity of the ſemi-Parabola AEB.

COROLLARY I.

274. IF the Exponent m be negative, then will $x=y^{-m}$ be the Equation of all kinds of Hyperbolas SMV with regard to the aſymptotes AR, AT, and $\frac{1-m}{1-2m}x$, $\frac{1-m}{4-2m}y$, will then expreſs the diſtances of the Center of gravity of the ſpace ARSMP when $1>2m$, or of the ſpace PMVT when $1<2m$, from the aſymptote AR. Fig. 152.

When $m=1$, then will $x=y^{-1}$ be the Equation of the common Hyperbola, and both the diſtances $\frac{1-m}{1-2m}x$, $\frac{1-m}{4-2m}y$, will become infinite in this caſe. *viz.* the firſt $\frac{1-m}{1-2m}x=\frac{0}{-1}x$, is that of the ſpace PMVT from the aſymptote AR, and the ſecond $\frac{1-m}{4-2m}y=\frac{0}{1}y$, that of the ſpace ARSMP from the aſymptote AT.

When $1=2m$, then the diſtance $\frac{1-m}{1-2m}x=\frac{1}{0}x$ of the Center of gravity of the ſpace ARSMP from the aſymptote AR is $=0$; and when $2=m$, then the diſtance $\frac{1-m}{4-2m}=\frac{-1}{0}y$ of the Center of gravity of the ſpace PMVT from the aſymptote AT, is $=0$.

EXAMPLE III.

275. TO *find the* Diſtance *of the Center of gravity of any ſpace as* BCMP *terminated by two lines parallel to the aſymptote* AR, *in the common Hyperbola, from that aſymptote.*

Let $AB=BC=1$, $BP=x$, $PM=y$, then will $\frac{1}{1+x}=y$ be of the Equation; $\frac{\dot{x}}{1+x}$ the fluxion of the Weights, and $\frac{x\dot{x}}{1+x}$ that the *Momentums.* Now if the fluent of $\frac{\dot{x}}{1+x}$, or the ſpace BCMP be $=z$, then will $x-z$ be the fluent of $\frac{x\dot{x}}{1+x}$ or the ſum of the *Momentums*, and $\frac{x-z}{z}$ the Expreſſion of the diſtance required. For $\dot{z}=\frac{\dot{x}}{1+x}$, and $\dot{x}-\dot{z}=\dot{x}-\frac{\dot{x}}{1+x}=\frac{x\dot{x}}{1+x}$.

EX-

EXAMPLE. IV.

276. TO *find the* Diſtance *of the Center of gravity of the Solid generated by the rotation of the ſemi-Parabola* ABE, *about the diameter* AE, *from the vertex* A. FIG. 165.

We ſhall have * $\frac{mc}{2r} \times \dot{y} y^{m+1}$ for the fluxion of the Weights, * *Art*, 236. and therefore $\frac{mc}{2r} \times \dot{y} y^{2m+1}$ will be that of the *Momentums*, becauſe $x = y^{m}$; whoſe fluent $\frac{m}{2m+2} \times \frac{c}{2r} y^{2m+2}$ divided by the ſum of the Weights $\frac{m}{m+2} \times \frac{c}{2r} y^{m+2}$, gives $\frac{m+2}{2m+2} y^{m} = \frac{m+2}{2m+2} x$ for the Expreſſion of the diſtance required.

When the Exponent m is negative, then $\frac{2-m}{2-2m} x$ will be the FIG. 152. Expreſſion of the diſtance of the Center of gravity of any Solid generated by the Hyperbolical Space ARSMP, about the aſymptote AT, from the other aſymptote AR when that Expreſſion is poſitive, or of that generated by the Space PMVT when it is negative.

When $m = 2$, that Expreſſion will become $= \frac{0}{-2} x$, and when $m = 1$, it will become $\frac{1}{0} x$, that is, infinite.

EXAMPLE. V.

277. TO *find the* Diſtance *of the Center of gravity of the circular arch* MAM, *from the line* TCT, *paſſing thro' the center* C *and drawn parallel to the chord* MM. FIG. 166.

It is evident, that the ſaid Center is ſome where in the Radius CA which biſects that arch: therefore, if the line m m, parallel to MM, meets that arch in m, m, and interſects the radius CA in p, and $Cp = x$, $pm = y$, the arch $Am = z$; then will $2\dot{z}$ be the fluxion

* Art. 164, 165. fluxion of the Weights, and $2\dot{z}x=$\*$2a\dot{y}$ the fluxion of the *Momentums*; and conſequently $\frac{ay}{z}$ will be the Expreſſion of the diſtance required of any arch as mm.

COROLLARY.

278. HEnce the arch MAM is to the chord MM, as the radius CA is to the diſtance of the Center of gravity of that arch from the center C; and therefore the ſemi-circumference is to its diameter, as the radius is to the diſtance of its Center of gravity from the center C.

EXAMPLE VI.

279. TO *find the* Diſtance *of the Center of gravity of the Sector* CMM, *from the center* C.

From the center C deſcribe any circular arch NBN, and draw Chord NQN, interſecting CA in Q: then if CN=CB=x, CA=a, MM=b, the arch MAM=c, the chord NN will be $=\frac{bx}{a}$, and the arch NBN $=\frac{cx}{a}$, as likewiſe* $\frac{\text{CN}\times\text{NN}}{\text{NBN}}=\frac{bx}{c}$ will be the diſtance of the Center of gravity of the arch NBN: and therefore $\frac{cx\dot{x}}{a}$ will be the fluxion of the weights; and becauſe the product of the weights multiplied by the diſtance of the Center of gravity * is equal to the *Momentums*, $\frac{bx}{c}\times\frac{cx\dot{x}}{a}=\frac{bxx\dot{x}}{a}$ will be the fixion of the *Momentums*; whoſe fluent $\frac{bx^3}{3a}$ divided by the weights, $\frac{cxx}{2a}$ gives $\frac{2bx}{3c}$; and when $a=x$, $\frac{2ab}{3c}$ will be the Expreſſion of the diſtance required.

* Art. 277.

* Art. 270.

COROLLARY.

280. HEnce, the arch MAM is to its chord MM, as $\frac{2}{3}$AC is to the diſtance of the Center of gravity of the Sector CMM from the center C: and the ſemi-circumference is to the diameter, as $\frac{2}{3}$AC is to the diſtance of the Center of gravity of the ſemi-circle, from the center C.

EXAMPLE VII.

281. TO *find the* Diſtance *of the Center of gravity of an Elliptical or Hyperbolical Space as* CAMQ, *terminated by the ſemi-ordinate* MQ *to the ſecond diameter* CB, *from the firſt* CA. FIG. 153, 154.

Becauſe* $\pm\frac{aa}{bb}yy=aa-uu$, we ſhall have $u=\frac{a}{b}\sqrt{bb\mp yy}$, * *Art.* 1. and* $(u\dot{y}=)\frac{a\dot{y}}{b}\sqrt{bb\mp yy}$ for the fluxion of the Weights, and * *Art.* 230. $\frac{ay\dot{y}}{b}\sqrt{bb\mp yy}$ for that of the *Momentums*; whoſe compleat fluent will be $\mp\frac{a}{3b}\times\overline{bb\mp yy}^{\frac{3}{2}}\pm\frac{abb}{3}$. And if the fluent of $\frac{a\dot{y}}{b}\sqrt{bb\mp yy}$, that is, if the ſpace CAMQ be $=z$; then will $\mp\frac{a}{3bz}\times\overline{bb\mp yy}^{\frac{3}{2}}\pm\frac{abb}{3z}$ be the Expreſſion of the diſtance required.

EXAMPLE VIII.

282. TO *find the* Diſtance *of the Center of gravity of the Solid, generated by the rotation of the ſpace* CAMQ *about the diameter* CB, *from the diameter* CA.

* *Art.* 233. We ſhall have * $\left(\frac{cu\dot{y}}{2r}=\right)\frac{caa}{2rbb}\times\overline{bb\dot{y}\mp yy\dot{y}}$ for the fluxion of the weights, and $\frac{caa}{2rbb}\times\overline{bby\dot{y}\mp y^3\dot{y}}$ for that of the *Momentums*; whoſe fluent $\frac{caa}{2rbb}\times\overline{\tfrac{1}{2}bbyy\mp\tfrac{1}{4}y^4}$ divided by the ſum of the Weights $\frac{caa}{2rbb}\times\overline{bby\mp\tfrac{1}{3}y^3}$, will give $\frac{1}{4}\times\frac{2bby\mp y^3}{3bb\mp yy}$ for the Expreſſion of the Diſtance required. When $y=b$, the ſaid Diſtance will become $=\frac{1}{8}b$ in the Ellipſis and Circle, and $\frac{9}{16}b$ in the Hyperbola.

EXAMPLE IX.

Fig. 157. 283. TO *find the* Diſtance *of the Center of gravity of the Surface, generated by the circular arch* BM *revolving about the diameter* CA, *from the center* C.

* *Art.* 235. If CP$=u$, the circumference $=c$, then will * $c\dot{u}$ be the fluxion of the Weights, and $cu\dot{u}$ that of the *Momentums*; whoſe fluent $\frac{cuu}{2}$ divided by the Weights cu, gives $\frac{u}{2}$ for the Expreſſion of the Diſtance required.

EXAMPLE X.

Fig. 156. 284. TO *find the* Diſtance *of the Center of gravity of the part* bNMP *of the* Ungula, *from the line touching the circular baſe at* b.

* *Art.* 257. We ſhall have* $\frac{b}{2a}\times\overline{2ax\dot{x}-xx\dot{x}}$ for the fluxion of the Weights, and $\frac{b}{2a}\times\overline{2axx\dot{x}-x^3\dot{x}}$ for that of the *Momentums*; whoſe fluent divided by the Weights, will give $\frac{8ax-3xx}{12a-4x}$ for the Diſtance required.

Becauſe the line drawn from the Center of gravity of any of the ſimilar triangles PMN, CAD, parallel to the ſide NM or DA, will meet the baſe PM or CA of that triangle, at a diſtance equal to * $\frac{2}{3}$PM or $\frac{2}{3}$ CA: it is evident, that if $\frac{b}{2a} \times \overline{2ax\dot{x} - xx\dot{x}}$, be multiplied by ($\frac{2}{3}$ PM $=$) $\frac{2}{3}\sqrt{2ax - xx}$, we ſhall have $\frac{b\dot{x}}{3a} \times \overline{2ax - xx}^{\frac{1}{2}}$ for the fluxion of the *Momentums.* Now if the fluent of $\dot{x}\sqrt{2ax - xx}$ or the circular ſpace bMP, be $= A$; then the fluent of $\frac{b\dot{x}}{3a} \times \overline{2ax - xx}^{\frac{1}{2}}$, will be $\frac{b}{3a} \times$ into $\frac{3aa}{4}A - \frac{\overline{a-x}}{4} \times \overline{2ax - xx}^{\frac{1}{2}}$, and the ſum of the Weights will be $= \frac{b}{2a} \times \overline{axx - \frac{1}{3}x^3}$. Therefore when $x = a$ (bP$=$bC); then will $\frac{3A}{4a}$ be the diſtance required of half the Solid bAC; and when $x = 2a$, we ſhall have $\frac{3A}{8a}$ for the diſtance of the Center of gravity of the whole Solid, from the diameter bB. Or if the ſemi-circular circumference bAB be $= c$, then will A be $= \frac{ac}{4}$ in the firſt caſe, and $A = \frac{ac}{2}$ in the ſecond: and ſo $\frac{3c}{16}$ will be the Expreſſion of the diſtance required in both caſes.

* *Art.* 271.

SCHOLIUM.

We may obſerve by the by, a noted Property of the Center of gravity which is, that the Solid generated by the rotation of a plane figure, or the Surface deſcribed by the rotation of any line, about an axis, is always equal to the product of the generating quantity multiplied by the circumference, deſcribed by its Center of gravity in the ſame time. *For if the plane figure* AMP *revolves about*

FIG. 151.

about the axis AP (x); *then if* PM$=y$, *the fluent of* $\frac{1}{2}yy\dot{x}$ *divided by the fluent of* $y\dot{x}$, *will give the Expreſſion of the diſtance of the Center of gravity from the axis* AP *of the Surface* AMP. *Now this Expreſſion being multiplied into* $\frac{c}{r}$, *will give* $S:\frac{cyy\dot{x}}{2ry\dot{x}}$ *for the circumference deſcribed by the Center of gravity: and therefore that circumference multiplied by* $S:y\dot{x}$, *that is, by the generating plane, will give* $S:\frac{cyy\dot{x}}{2r}$, *for the* * *content of the Solid generated in that revolution.*

* *Art.* 233.

SECTION IV.

The Uſe of Fluxions in finding the Centers of Oſcillation *and* Percuſſion.

DEFINITIONS.

1. A *compound Pendulum*, conſiſts of ſeveral Bodies or Weights fixed together: and a *ſimple Pendulum*, conſiſts of a ſingle Body or Weight conſidered as a point.

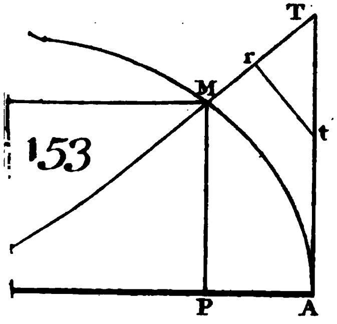
2. The line paſſing thro' the common Center of gravity of a compound Pendulum and thro' the point of Suſpenſion, is called the *Axis* of the Pendulum.

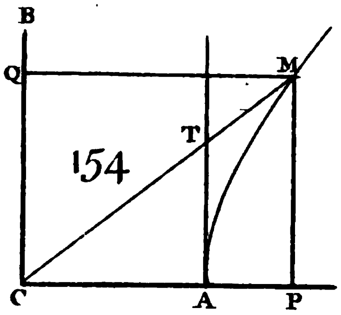
3. The line paſſing thro' the point of Suſpenſion and about which the Pendulum Oſcillates, is called the *Axis* of *Ocillation.*

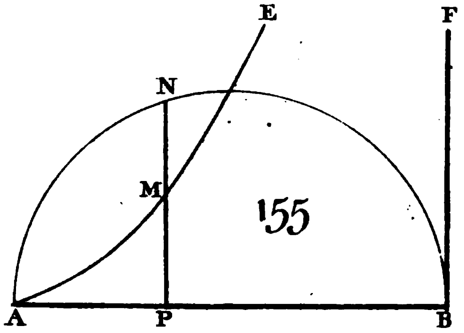
4. Two Pendulums are ſaid to be *Iſocronous* when they perform an equal number of Vibrations in the ſame given time.

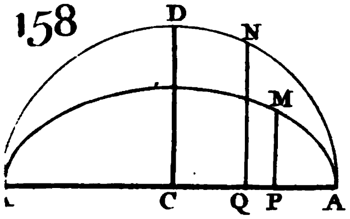
5. The *Center* of *Oſcillation*, is a point in the axis of a compound Pendulum, whoſe diſtance from the point of Suſpenſion is equal to the lengths of a ſimple Pendulum, Iſocronous to the former.

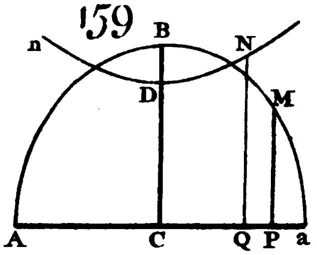
6. If you conceive ſeveral particles, each multiplied by its reſpective velocity, as ſo many Weights applied at different diſtances from the axis of Oſcillation; then the power which they have

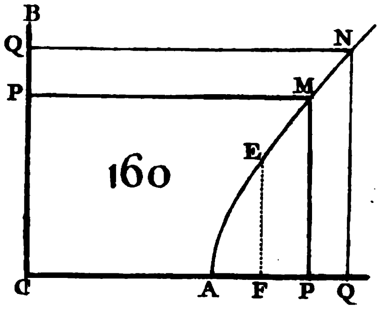

153

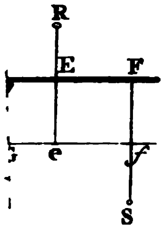

154

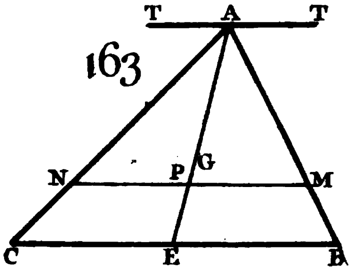

155

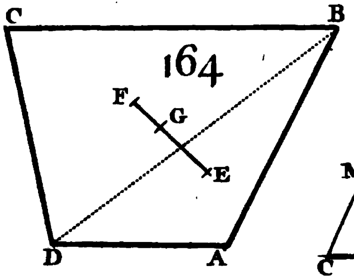

158

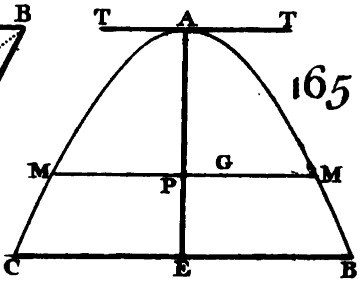

159

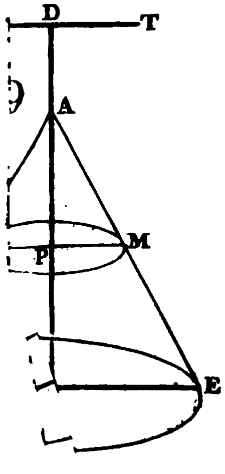

160

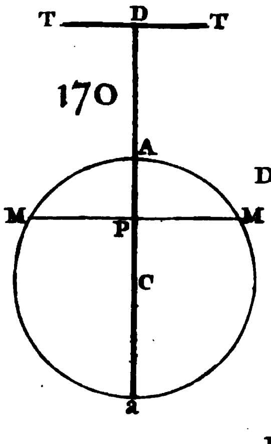


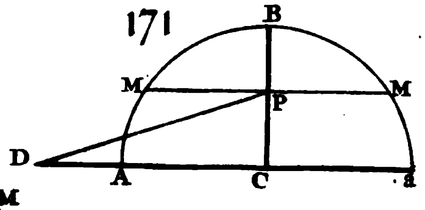

163

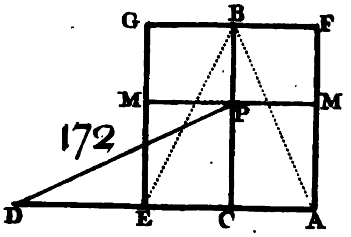

164

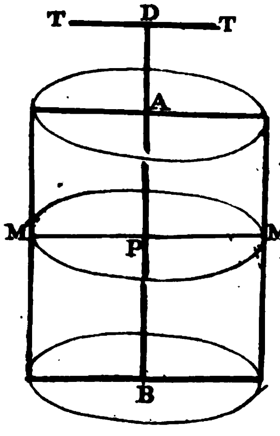

165


9


170


171


172


173

have to compel the Pendulum to vibrate, with regard to their diſtances from the axis of Oſcillation, is called their *Force*.

COROLLARY.

285. IF A, B, C, *&c.* are ſo many particles of a Pendulum, revolving about the axis D perpendicular to the plane DABC, ſo that DA, DB, DC, expreſs their reſpective diſtances from the axis of Oſcillation; and if in the axis DO of the Pendulum, the point O be taken ſuch, that all the *Momentums*, or products of all the particles each multiplied by its velocity, were united in that point, the Pendulum would perform an equal number of vibrations in the ſame time as a ſimple Pendulum Iſoronous to this; then the point O will be the Center of Oſcillation of the Pendulum. Fig. 167.

GENERAL PROBLEM.

286. *TO find the* Diſtance *of the Center of Oſcillation* O *of a Pendulum* DABC, *from the point of Suſpenſion.*

Draw the linds AP, BQ, CR, perpendicular to DO, and the line A*a* perpendicular to the diſtance DA; then becauſe A*a*, will be the direction of the velocity of the particle A, in the Vibration of the Pendulum, it will be the ſame thing, whether that particle be fixed in *a* or A, provided it has the ſame velocity, and the ſame direction: and as the velocity of each particle, is as its diſtance from the axis of Oſcillation; if A*a* be produced towards b, ſo that $ab = DA$, the velocity DA or *a*b of the particle A, conſidered as fixed in *a*, will be equivalent to the two *ac*, *c*b; the firſt perpendicular and the laſt parallel to the axis DO of the Pendulum, and *ac* will be the velocity with which the particle A turns the axis; therefore $A \times ac$ will be its *Momentum*, and the *Momentum* multiplied by by itsdiſtance D*a*, will give $A \times Da \times ac$ for its force. But becauſe $DA = ab$, and the triangles DAP, *abc* are ſimilar, ac will be $= DP$; and by reaſon of the ſimilar triangles DPA, DA*a*, $\overline{DA}^2$ will be $= DP \times Da = Da \times ac$: conſequently $A \times DP$ will be the *Momentum* of the particle A, and $A \times \overline{DA}^2$ its force.

In like manner it may be proved, that $B \times DQ$, $C \times DR$, are the *Momentums* of the particles B, C; and $B \times \overline{DB}^2$, $C \times \overline{DC}^2$, their forces.

But the ſum of the *Momentums* multiplied by the diſtance DO, ought to be * equal to the ſum of the forces of all the particles. * *Art.* 285. Therefore $A \times \overline{DA}^2 + B \times \overline{DB}^2 + C \times \overline{DC}^2 +$ &c., $= DO \times$ into $A \times DP + B \times DQ + C \times DR +$ &c., conſequently

$$DO = \frac{A \times \overline{DA}^2 + B \times \overline{DB}^2 + C \times \overline{DC}^2 + \&c.}{A \times DP + B \times DQ + C \times DR + \&c.}.$$

If the number of particles were infinite, it may in like manner be provided, that the ſum of all the particles each multiplied by the ſquare of its reſpective diſtance from the axis of Oſcillation, divided by the ſum of their *Momentums,* will be equal to the diſtance of the Center of Oſcillation required.

If ſome particles of the Pendulum were placed on the other ſide of the axis of Oſcillation, with reſpect to the reſt; then the difference of the ſum of the forces of thoſe on one ſide, to the ſum of thoſe on the other, divided by the ſum of all the *Momentums,* will be equal to the diſtance required. This is a natural conſequence from what has been ſaid above, and may be eaſily proved.

A General RULE for finding the *Diſtance* of the Center of Oſcillation of a Pendulum from *the axis* of Oſcillation.

Find the Sum *of all the products of the particles, each multiplied by the ſquare of its diſtance from the axis of Oſcillation, if they are all placed on one ſide of the axis, or the* Difference *of the like products, if placed on both ſides, and divide that Sum or Difference by the* Sum *of all the Momentums, the* Quotient *will expreſs the* Diſtance *required.*

EXAMPLE I.

287. TO *find the* Diſtance *of the Center of Oſcillation of a Triangle* ABC *revolving about the axis* TAT, *which paſſes thro' the vertex* A *and is parallel to the baſe* BC. Fig. 163.

We have * $\frac{bxx\dot{x}}{a}$ for the fluxion of the *Momentums*, and $\frac{bx\dot{x}}{a} \times xx = \frac{bx^3\dot{x}}{a}$ for the fluxion of the Forces; whoſe Fluent or ſum of the Forces $\frac{bx^4}{4a}$ divided by the *Momentums* $\frac{bx^3}{3a}$, will give $\frac{1}{4}x = \frac{1}{4}$AE, for the Diſtance required.

* Art. 271.

EXAMPLE II.

288. TO *find the* Diſtance *of the Center of Oſcillation of the Parabola* BAC *revolving about the Tangent* TAT *parallel to the baſe* BC. Fig. 164.

If $x = y^m$ be the Equation, we ſhall have $(yx\dot{x} =)$ $m\dot{y}y^{2m}$ for the Fluxion of the *Momentums*, and $(y\dot{x} \times xx =)$ $m\dot{y}y^{3m}$ for the Fluxion of the Forces; the fluent of which or the ſum of the Forces $\frac{m}{1+3m}y^{3m+1}$, divided by the ſum of the *Momentums* $\frac{m}{1+2m}y^{2m+1}$, gives $\frac{1+2m}{1+3m}y^m = \frac{1+2m}{1+3m}x$, for rhe Diſtance required.

If m be negative; then $x = y^{-m}$ will be the Equation of all kinds of Hyperbolas with reſpect to the aſymptotes, and $\frac{1-2m}{1-3m}x$ will then be the Expreſſion of the Diſtance required. Fig. 154.

EX-

EXAMPLE III.

289. *LET the Parabola* BAC *revolve about its baſe* BC.

Fig. 165 If AE$=a$, we ſhall have (EP$\times y\dot{x}=$) $\overline{a-y^m}\times m\dot{y}y^m$ (becauſe $x=y^m$) for the fluxion of the *Momentums*, and $\overline{a-y^m}^2\times m\dot{y}y^m$ for that of the Forces; the fluent of which or the ſum of the Forces, will be $\frac{aam}{1+m}y^{m+1}-\frac{2am}{1+2m}y^{2m+1}+\frac{m}{1+3m}y^{3m+1}$, and the *Momentums* will be $\frac{am}{1+m}y^{m+1}-\frac{m}{1+2m}y^{2m+1}$; and when $x=y^m=a$, the Forces divided by the *Momentums*, will give $\frac{2am}{3m+1}$ for the Diſtances required.

EXAMPLE IV.

Fig. 170. 290. *LET the Circle* AMaM *revolve about the line* TT *perpendicular to the diameter* aA *produced, and parallel to the plane of the Circle.*

If AD$=d$, AP$=x$, Aa $=2a$; then the ſemi-ordinate PM will be $=\sqrt{2ax-xx}$, and $\overline{d+x}\times\dot{x}\sqrt{2ax-xx}$ will be the Fluxion of the *Momentums*, and $\overline{d+x}^2\times\dot{x}\sqrt{2ax-xx}$, that of the Forces: therefore if the fluent of $\dot{x}\sqrt{2ax-xx}$ or the circular ſpace AMP be $=z$; then will $ddz+\frac{5aaz}{4}+2adz-\frac{2}{3}dy^3-\frac{1}{4}xy^3-\frac{5}{12}ay^3$ be the ſum of the Forces, and $dz+az-\frac{1}{3}y^3$ the ſum of the *Momentums*, becauſe $y=\sqrt{2ax-xx}$; and when $x=a$; y will be$=0$, and $\frac{4dd+5aa+8ad}{4d+4a}$ will expreſs the diſtance required.

LEM-

LEMMA I.

291. *LET* ABa *be a ſemi-circle,* CB *a radius perpendicular to the diameter* Aa, *and* MPM *a parallel to* Aa *interſecting* CB *in* P: *then if from any given point* D *in the diameter* Aa, *a line* DP *be drawn: the ſum of all the products of the ſquares* $\overline{DP}^2$ *each multiplied by the correſpondent particle* M *of the circular quadrant* AMB, *will be equal to* $\overline{DC}^2 + \frac{1}{2}\overline{AC}^2 \times$ *into the arch* AMB$=c$. FIG. 171.

For if AC$=a$, CD$=d$, CP$=x$, and the arch AM$=z$; then will * $\dot{z} = \frac{a\dot{x}}{\sqrt{aa-xx}}$, and $\overline{DP}^2 = dd+xx$. Therefore $\dot{z} \times \overline{DP}^2 =$ * Art. 250. $dd\dot{z} + xx\dot{z}$; whoſe fluent will be $ddz + \frac{aaz}{2} - \frac{ax}{2}\sqrt{aa-xx}$; and when $x=a$, then will $aa-xx=0$, $z=c$; and conſequently $ddc + \frac{1}{2}aac$ will be the product deſired.

LEMMA II.

292. *THE Sum of the products of all the Squares* $\overline{DP}^2$ *each multiplied by the correſpondent fluxion* ($y\dot{x}=$) $\dot{x}\sqrt{aa-xx}$ *of of the Space* CPMA, *will be* $= ddn + \frac{1}{4}aan$, *ſuppoſing* n *equal to the area of the quadrant.*

Let $\dot{v} = \dot{x}\sqrt{aa-xx}$, then the fluent of ($\dot{v} \times \overline{DP}^2 =$) $dd\dot{v} + xx\dot{v}$ will be $ddv + \frac{1}{4}aav - \frac{1}{4}x \times \overline{aa-xx}^{\frac{3}{2}}$, which becomes equal to $ddn + \frac{1}{4}aan$, when $a=x$; becauſe $v=n$, and $aa-xx=0$.

EXAMPLE V.

FIG. 170. 293. LET *the Surface of the Sphere* AMaM *revolve about the line* TT.

If AD $=d$, AP$=x$, PM$=y$, Aa $=2a$, the circumference AMaM$=c$, and the arch AM$=z$; then will $\frac{cy}{a}$ be the circumference whoſe radius is PM (y), and $\overline{DP}^2=\overline{d+x}^2$. Therefore the product of all the particles of the circumference $\frac{cy}{a}$, each multiplied into the Square of its diſtance from the axis of Oſcillation,

* *Art.* 291. will be* $=dd+2dx+ax+\frac{1}{2}xx \times$ into $\frac{cy}{a}$, becauſe $yy=2ax-xx$;

* *Art.* 250. which being multiplied by the fluxion* $\dot{z}=\frac{a\dot{x}}{y}$ of the circular arch AM, gives $dd+2dx+ax+\frac{1}{2}xx \times$ into $c\dot{x}$ for the fluxion of the Forces; whoſe fluent $ddx+dxx+\frac{1}{2}axx+\frac{1}{6}x^3 \times$ into c, will become $2add+4ad+\frac{12}{3}a^3 \times$ into c, when $x=2a$. Now this being divided by the product of the Surface of the Sphere$=2ac$, multiplied into the diſtance CD$=d+a$ of the Center of gravity C of that Surface from the point of ſuſpenſion D, will give $\frac{3dd+6ad+5aa}{3d+3a}$ for the Diſtance required.

EXAMPLE VI.

294. LET *the Sphere* AMaM *revolve about the axis* TT.

Becauſe $yy=2ax-xx$, we ſhall have $\left(\frac{cyy}{2a}=\right)cx-\frac{cxx}{2a}$ for

* *Art.* 292. the Circle deſcribed by PM, and becauſe $\overline{DP}^2=\overline{d+x}^2$*, $dd+2dx+\frac{1}{2}ax+\frac{1}{4}xx \times$ into $cx-\frac{cxx}{2a}$ will be the product of all the particles of the circle MM, each multiplied into the Square

of its diſtance from the axis of Oſcillation; and therefore $dd+2dx+\frac{1}{2}ax+\frac{1}{4}xx \times$ into $cx\dot{x}-\frac{cxx\dot{x}}{2a}$ will be the fluxion of the Forces; whoſe fluent $\frac{1}{2}dd+\frac{2}{3}dx+\frac{1}{6}ax+\frac{1}{8}xx-\frac{ddx}{6a}-\frac{dxx}{4a}-\frac{3x^3}{40a}\times$ into cxx, will become $A=\frac{1}{3}dd+\frac{5}{12}ad+\frac{13}{60}aa\times$into aac, when $x=a$, or $B=\frac{1}{6}dd+\frac{1}{3}ad+\frac{7}{30}aa\times$into $4aac$, when $x=2a$. Theſe values being divided, the firſt A by the product of the content of half the Sphere $\frac{aac}{3}$ multiplied into the diſtance of its Center of gravity * $d+\frac{5}{8}a$; and the ſecond B, by the product of the whole Sphere $\frac{2aac}{3}$ multiplied into the diſtance of its Center of gravity $CD=d+a$, that is, the firſt by $\frac{aac}{3}\times\overline{d+\frac{5}{8}a}$ and the laſt by $\frac{2aac}{3}\times\overline{d+a}$ we ſhall have A, $\frac{40dd+50ad+26aa}{40d+25a}$, and B, $\frac{5dd+10ad+7aa}{5a+5d}$ for the Expreſſions of the diſtances of the Centers of Oſcillation of half the Sphere and of the whole, from the point of ſuſpenſion.

* Art. 282.

EXAMPLE VII.

295. *LET the Cylinder* AB *revolve about the line* TT *perpendicular to its axis* AB. FIG. 173.

Suppoſe the Cylinder to be cut by a plane MPM perpendicular to its axis, and call the radius $PM=a$, its circumference $=c$, $AP=x$, $AD=d$; then will $\overline{d+x}\times\frac{ac\dot{x}}{2}$ be the fluxion of the *Momentums*, and $dd+2dx+xx+\frac{1}{4}aa\times$ into $\frac{ac\dot{x}}{2}$ that of the Forces; whoſe fluent divided by the *Momentums* gives

12d+

$\frac{12dd+12xd+4xx+3aa}{12d+6x}$ for the diſtance required of any part as AMM.

EXAMPLE VIII.

FIG. 169. 296. *LET the right Cone* EAF *revolve about the line* TT *perpendicular to its axis* CA.

Suppoſe the Cone to be cut by a plane MPM parallel to its baſe, and make DA $=d$, AP $=x$, PM $=y$, AC $=a$, CE $=r$, and the circumference of the baſe $=c$; then the circle deſcribed by the radius PM will be $=\frac{cyy}{2r}$, or becauſe AC : CF : : AP : PM $=y=\frac{rx}{a}$, $\overline{d+x}\times\frac{crxx\dot{x}}{2aa}$ will be the Fluxion of the *Momentums*, and $dd+2dx+xx+\frac{rr}{4aa}xx\times$ into $\frac{crxx\dot{x}}{2aa}$ the Fluxion of the Forces; the fluent of which $\frac{1}{3}dd+\frac{1}{2}dx+\frac{1}{5}xx+\times\frac{rrxx}{20aa}\times$ into $\frac{crx^3}{2aa}$ divided by the *Momentums* $\overline{\frac{1}{3}d+\frac{1}{4}x}\times\frac{crx^3}{2aa}$ will give $\frac{20dd+30ad+12aa+3rr}{20d+15a}$ when $x=a$, for the Expreſſion of the diſtance required.

LEMMA III.

FIG. 172. 297. *IF the Rectangle* AG *be biſected by the line* BC *parallel to its ſides* AF, EG, *and if* MM *parallel to* AE *interſects* CB *in* P, *and from that point a line* PD *be drawn to any given point* D *in the baſe* AE; *then the ſum of all the products of the Squares* $\overline{DP}^2$ *of the lines drawn from the point* D *to all the points of the line* CB, *each multiplied by the correſpondent line* MM, *will be equal to* $\overline{DC}^2+\frac{1}{3}\overline{BC}^2\times$ *into the rectangle* AG.

For if DC $= d$, MM $= a$, BC $= b$, PC $= x$; then will $a\dot{x}\times\overline{dd+xx}$ be the Fluxion of that quantity; whoſe fluent $ax\times\overline{dd+\frac{1}{3}xx}$ will become $ab\times\overline{dd+\frac{1}{3}bb}$, when $x=b$.

LEMMA IV.

298. *THE ſum of all the products of the lines* mm *drawn in a Triangle* ABE *parallel to its baſe* AE, *each multiplied by the Square of the correſpondent line* PD, will be $=\overline{\text{DC}}^2+\frac{1}{6}\overline{\text{BC}}^2$ × into the area of that Triangle.

For becauſe BC : AE : : BP$(b-x)$: mm $= \frac{ab-ax}{b}$, we ſhall have $\frac{ab\dot{x}-ax\dot{x}}{b}$ × into $\overline{dd+xx}$ for the Fluxion of that quantity; whoſe Fluent $ax\times$ into $dd+\frac{1}{3}xx-\frac{ddx}{2b}-\frac{x^3}{4b}$ will become $ab\times$ into $\frac{1}{2}dd+\frac{1}{12}bb=\frac{ab}{2}\times\overline{dd+\frac{1}{6}bb}$, when $x=b$.

EXAMPLE IX.

299. *LET the right Pyramid* EAL *revolve about the line* TT *perpendicular to its axis* CA, *and parallel to one of the ſides* FL *of its baſe.* FIG: 168.

If the Pyramid be cut by a plane MM parallel to its baſe, and the line MM be drawn in that plane parallel to the axis TT of revolution, interſecting the axis of the Pyramid in P: then if AP$=x$, MM$=y$ AD$=d$, AC$=a$, FL or EG$=b$; we ſhall have MM$=y=\frac{bx}{a}$; and therefore $\overline{d+x}\times\frac{bbxx\dot{x}}{aa}$ will be the

Fluxion

* *Art.* 297. Fluxion of the *Momentums*, and* $\overline{dd+2dx+xx+\frac{bb}{3aa}xx}\times$ into $\frac{bbxx\dot{x}}{aa}$ the Fluxion of the Forces; whoſe fluent will be $\overline{\frac{1}{3}dd+\frac{1}{2}dx+\frac{1}{5}xx+\frac{bb}{15aa}xx}\times$ into $\frac{bbx^3}{aa}$, and that of the *Momentums* will be $\overline{\frac{1}{3}d+\frac{1}{4}x}\times\frac{bbx^3}{aa}$; and when $x=a$, we ſhall have $\frac{20dd+30ad+12aa+4bb}{20d+15a}$ for the Expreſſion of the diſtance required.

EXAMPLE X.

300. NOW *let the Pyramid revolve about the line* TT *parallel to one of its diagonals* EL.

We ſhall have $\overline{d+x}\times\frac{bbxx\dot{x}}{aa}$ for the Fluxion of the *Momentums*, and* $\overline{dd+2dx+xx+\frac{bb}{6aa}xx}\times$ into $\frac{bbxx\dot{x}}{aa}$ for the Fluxion of the Forces; whoſe Fluent will be $\overline{\frac{1}{3}dd+\frac{1}{2}dx+\frac{1}{5}xx+\frac{bb}{30aa}xx}\times$ into $\frac{bbx^3}{aa}$; and that of the *Momentums* will be $\overline{\frac{1}{3}d+\frac{1}{4}x}\times\frac{bbx^3}{aa}$. Which gives $\frac{20dd+30ad+12aa+2bb}{20d+15a}$, when $x=a$, for the diſtance required.

* *Art.* 298.

DEFINITION.

The *Center* of *Percuſſion* of a moving Body, is that point of the ſame, wherein, if in its motion it meets any obſtacle, it will ſtrike it with the greateſt Force poſſible.

PRO-

PROBLEM.

301. *TO find the* Center *of perćuſſion* O *of two Bodies* A, B, *conſidered as points placed any how, and fixed together in a given plane* DAB. Fig. 174.

Let the axis of revolution D be perpendicular to the Scheme, and from the points A, B, draw AP, BQ, perpendicular to the line DO, drawn any where in that plane, and A*a*, Bb, perpendicular to the lines AD, BD expreſſing the diſtances of thoſe Bodies from the axis of revolution, and from the Center O the perpendiculars Oc, Od, to the lines A*a*, Bb: then it is evident, that if the *Momentums* of the Bodies A and B, are reciprocally as the diſtances Od, Oc, of their directions from the point O, the Figure DAB, will ſtrike any obſtacle which meets the ſame in the point O perpendicular to the plane DO, with the greateſt Force poſſible; ſince the quantities of the Forces, being equal on both ſides of that point; it will ſtrike the obſtacle in the ſame manner as if the whole Force was united in the ſame.

Therefore, if $DA=a$, $DP=b$, $DB=c$, $DQ=d$, $DO=n$; then the ſimilar triangles DPA, DA*a*, Oc*a* and DQB, DBb, Odb, will give, $DP : DA :: DA : Da = \frac{aa}{b}$, and $DQ : DB :: DB : Db = \frac{cc}{d}$; and ſo $aO = n - \frac{aa}{b}$, $Ob = \frac{cc}{d} - n$. Likewiſe $DA : DP :: Oa : Oc = \frac{bn-aa}{a}$, and $DB : DQ :: Ob : Od = \frac{cc-dn}{c}$. Conſequently if p, q, expreſs the weights of the Bodies A and B; we ſhall have $(DA \times p)ap : (DB \times q)cq :: (Od) \frac{cc-dn}{c} : (Oc) \frac{bn-aa}{aa}$, or $DO = n = \frac{aap+ccq}{bp+dq}$. Which ſhews, that if the plane DO paſſes thro' the Center of gravity of the Bodies A and B, the Center of percuſſion O will be the ſame as the Center of *Oſcillation.*

SECTION V.

The Uſe of Fluxions in the Reſolution of a Select Collection of Phyſico-Mathematical Problems.

In the following *Problems,* we conſider planes as perfectly ſmooth; Bodies as moving in a non-reſiſting Medium; Engines without Friction, and Fluids as Bodies whoſe parts give way to any Force whatever acting upon them.

LEMMA I.

Fig. 175. 302. I*F three Powers,* P, Q, R, *are in equilibrium with each other; the three ſides* AB, AC, BC, *of a Triangle* (ABC) *which interſect their Directions at right angles* (*ſuppoſe in* G, H, O,) *will expreſs the ratio's of thoſe Powers reſpectively.*

Let the parallelogram OEAF be made of their directions. Now OE, AE (=OF) AO, expreſs the ratio's of thoſe powers P, Q, R (by Mechanics); and becauſe of the right angled ſimilar Triangles AOB, AGO, and AOC, AHO, the angles ABO, AOG will be equal as well as the angles ACO, OAE (=AOH). Therefore the Triangles BAC, OEA, are ſimilar; and ſince the ſides of the one expreſs the ratio's of thoſe powers, the ſides of the other will expreſs them alſo. *W. W. D.*

PROBLEM I.

Fig. 176. 303. L*ET either the interior or exterior Curve of a* Vault *or* Arch *be given, to find the other, ſuch that all the* Vouſſoirs *remain in equilibrium.*

Vaults may be either generated by the parallel motion of a plane AHLd, or by the circular motion of that plane about the vertical line CH as an axis. Hence if the interior Curve AcB be given, the exterior Curve HKL, will not be the ſame in both caſes. For by the firſt formation, the Vouſſoirs AHIb, bIKc, *&c.* become Priſms, which will be as the generating planes: and by the ſecond, the Vouſſoirs become a ſort of crowns, which will not be as their generating planes: and therefore the Curves will not be the ſame in both caſes.

Caſe I. *Of* Vaults *generated by a parallel a motion.*

.I Suppoſe the interior Curve AbcdB be given, and let all the joints Ib, Kc, Ld, *&c.* produced, meet in the ſame point C: and from the Centers of gravity, V, X, Y, of the Vouſſoirs, let the lines VR, XS, YT, be drawn at right angles to the horizontal line DQ, drawn at pleaſure, and interſecting the joints in E, F, G, *&c.* Now it is evident, that if DE repreſents the gravity of the Vouſſoir V, EF that of the Vouſſoir X, and FG that of of the Vouſſoir Y; the line CD will repreſent the Force with which the Vouſſoir V preſſes the Vertical joint HA, CE the Forces with which the Vouſſoirs V, X, preſs the joint Ib, and ſo on. For the ſide DE of the Triangle CDE, is perpendicular to the direction VR of the Center of gravity of the Vouſſoir V, and the ſides CD, CE perpendicular to the directions Vr, V*x* in which the Vouſſoir V preſſes the joints AH, Ib, and ſo of the reſt. Therefore, ſince the ſame line CE, expreſſes the Forces with which both the Vouſſoirs X, V, preſs each other in contrary directions, they will be in equilibrium. Likewiſe, as the ſame line CF, expreſſes the Forces with which the Vouſſoirs X, Y, preſs one another in contrary directions, they will be likewiſe in equilibrium, and ſo of the reſt. Therefore if the Curve HIK be ſuch, that the ſpace AHIb be always equal to the correſpondent Triangle CDE, it will be that required; becauſe the height CD of the Triangle is given, its area will be as the baſe DE.

Now becauſe the circular Sectors deſcribed by the radii CE, Cb, CI, in the ſame time, are as the Squares of thoſe radii, and ſince thoſe Sectors are * as the Fluxions of the Spaces CDE, CAb, CHI; we ſhall have $\overline{CE}^2 = \overline{CI}^2 - \overline{Cb}^2$, or $\overline{CI}^2 = \overline{CE}^2 + \overline{Cb}^2$.

* *Art.* 229.

When CI becomes =CH, Cb will become=CA, CE=CD, and $\overline{CD}^2=\overline{CH}^2-\overline{CA}^2$, therefore if the height CA and the thickneſs AH of the Vault are given, the line DQ will be given in poſition, and conſequently, if CE be taken ſucceſſively equal to given lines, the radius CI will be given alſo; by the help of which the exterior Curve HIKL may be deſcribed.

II. Suppoſe the joints produced, do not meet all in the ſame Fig. 177. point, but are always perpendicular to the interior Curve. Produce the joint MN taken at pleaſure, ſo as to meet the Evolute of the interior Curve in D, and from any point d of the vertical joint HA, draw the indefinite line dQ parallel to the horizon CB, and from C the line Ce parallel to the joint NM. Now it may be proved in the ſame manner as before, that if the ſide de of the Triangle Cde, expreſſes the gravity of the Vouſſoir AHMN, the two other ſides Cd, Ce will expreſs the Forces with which that Vouſſoir preſſes the joints AH, NM; and therefore, if the Area AHMN be equal to that of the correſpondent Triangle Cde, the Curve HM will be that required.

But becauſe the angular motion of the lines Ce, DM, is always equal by ſuppoſition, the circular Sectors deſcribed by the radii Ce, DN, DM, will be as the Squares of thoſe radii; and therefore $\overline{DM}^2-\overline{DN}^2=\overline{Ce}^2$, or $\overline{DM}^2=\overline{Ce}^2+\overline{CN}^2$. Which ſhews that the difference in the two *Caſes* is, that Cb expreſſes any radius drawn from a given Center in the former, and *DN* expreſſes any radius of evolution in the latter: conſequently if the interior Curve is given, the radius of evolution drawn from * Art. 194. any given point * may be found, and ſo the Curve HM may be deſcribed.

If ANB be an Elliptical quadrant, BC one of its Semi-axis, and CA the other; then if the part KN of the radius of evolution terminated by the axis CA, be $= c$, and the parameter of the * Art. 200. Semi-axis CB, $=p$; we ſhall have * DN $= \frac{c^3}{pp}$.

Caſe

Caſe II. *Of* VAULTS *generated by a* Circular Motion.

I. IF the Surface deſcribed by the line EF, expreſſes the gravity of the Vouſſoir generated by the rotation of the Space bIKc, about the axis CH, the Surface deſcribed by the line CE, or CF, in that rotation, will expreſs the Force with which the adjoining Vouſſoirs to the joint bI or cK, preſs each other in contrary directions. For if the line EF expreſſes the gravity of the generating plane bIKc, the lines CE, CF, will expreſs the Forces with which that plane preſſes the joints bI, cK: and therefore the Surface deſcribed by EF, will expreſs the gravity of the whole Solid generated by the figure bIKc, as likewiſe the Surfaces deſcribed by the lines CE, CF, will expreſs the preſſures of that Vouſſoir, againſt thoſe adjoining to it. The ſame thing may be proved with regard to any other Vouſſoir. Conſequently if the Curve HIK be ſuch, that the Solid generated by any Space bIKc, be always equal to the Solid generated by the correſpondent Triangle CEF in the ſame time, it will be that required. FIG. 176.

But becauſe the circular Sectors CEo, CNn, CMm, deſcribed by the radii CE, CN, CM, in the ſame time, are as the Squares of thoſe radii; and as the fluxions of the Solids generated by the rotation of the Spaces CDE, CAN, CHM, about the axis CH, are as thoſe circular Sectors, each multiplied by the circumference deſcribed by its center of gravity, or becauſe the circumferences are as their radii, if y, *x*, u, are the Centers of gravities of thoſe Sectors, and the lines yr, *x*q, up, are drawn at right angles to CA; the fluxions of thoſe Solids will be as $CEo \times yr$, $CNn \times xq$, $CMm \times up$. But ſince $Cy : yr :: Cx : xq :: Cu : up$; and the diſtances Cy, C*x*, Cu, are* as the radii, CE, CN, CM, thoſe fluxions will be as the cubes of the lines CE, CN, CM: Therefore $\overline{CM}^3 - \overline{CN}^3 = \overline{CE}^3$, or $\overline{CM}^3 = \overline{CE}^3 + \overline{CN}^3$. FIG. 178. * *Art.* 279.

When CM = CH, CN will become = CA, CE = CD, and $\overline{CD}^3 = \overline{CH}^3 - \overline{CA}^3$. Therefore if the thickneſs of the Vault at the vertex is given, the line DQ will be given in poſition, and conſequently the Curve HMZ may be deſcribed.

II. Suppoſe that all the joints produced, do not meet in the ſame point, but are perpendicular to the interior Curve. It is evident, that if we retain the ſame Conſtruction of the figure 177, by the ſame way of reaſoning as in the firſt caſe, we ſhall find that if $\overline{DM}^3=\overline{DN}^3+\overline{Ce}^3$; the Curve HMZ deſcribed by means of this Equation, will be that required.

COROLLARY I.

Fig. 176. 304. IN the parallel motion, if the interior Curve becomes a right line parallel to DQ; we ſhall always have CD : CA :: CE : Cb $= \frac{CA\times CE}{CD}$: and therefore $\overline{CI}^2=\overline{CE}^2+\overline{Cb}^2$ will become $CI=\frac{CE}{CD}\sqrt{\overline{CA}^2+\overline{CD}^2}$: and becauſe all the lines except CE, are given, the ratio of CE to CI will be given alſo, conſequently the exterior Curve will be a ſtrait line parallel to DQ *in this caſe.*
Fig. 178. And in the circular motion, if the interior Curve ANB becomes a right line; then will always be CD : CE :: CA : CN $= \frac{CA\times CE}{CD}$: and therefore CM $= \frac{CE}{CD}\sqrt[3]{\overline{CA}^3+\overline{CD}^3}$; which ſhews that the exterior Curve becomes likewiſe a right line.

A Conſtruction of ſome Vaults *generated by a Parallel Motion.*

Fig. 179. *Caſe* 1. Let the interior Curve AB be a circular Quadrant deſcribed from the center C, and with the radius CA, and ſuppoſe AH to be the bigneſs of the Vault at the vertex; then if BD be made =CH, and from the point D the indefinite line DQ be drawn at right-angles to CA; and after having drawn ſeveral radii CI, CK, CL, *&c.* interſecting the line DQ in E, F, G, *&c.* you make one of the legs of a right angle *a*cg, as c*a* equal to CA, and you take upon the other ce=CE, cf = CF, cg = CG, *&c*; then making always CI=*a*e, CK = *af*, CL=*ag*;

the Curve paſſing thro' the points H, I, K, L, &c. will be that required.

For becauſe CA = BC, and CH = BD, we ſhall have $\overline{CD}^2 = \overline{CH}^2 - \overline{CA}^2 = \overline{BD}^2 - \overline{CB}^2$. And ſince c*a* = CA, c*e* = CE, $\overline{ae}^2 = = \overline{CI}^2$ will be $= \overline{CE}^2 + \overline{CA}^2$.

Caſe 2. Let the interior Curve AB be a circular arch deſcribed Fig. 180.
from the center C, with the radius CB or CA, and let the part AH of the radius drawn from the center C thro' the vertex A, expreſs the thickneſs of the Vault in that place. Upon CH as a diameter deſcribe the ſemi-circular circumference HMC, make CM = CA, and CD = HM; and draw the indefinite right line DQ parallel to horizontal line CB: then after having taken upon one of the legs of a right angle *a*c*g*, c*a* = CA, and upon the other ce = CE, c*f* = CF, c*g* = CG, &c., you make always CI = *a*e, CK = *af*, CL = *ag*, &c., the Curve-line HIK, will be that required.

For becauſe CM = CA, and HM = DC, $\overline{DC}^2$ will be $= \overline{CH}^2 - \overline{CA}^2$: and ſince c*a* = CA and c*e* = CE, $\overline{ae}^2 = \overline{CI}^2$ will be $= \overline{CE}^2 + \overline{CA}^2$.

Caſe 3. Let the interior Curve ANB be an elliptical Quadrant, Fig. 177.
and the joints MN, &c, be perpendicular to the Ellipſis, and interſecting the axis CB in K, &c.; then if $\frac{\overline{CB}^2}{CA} = b$, and $Cd = \sqrt{\overline{b+AH}^2 - bb}$ and the indefinite line dQ be drawn perpendicular to AC; and after having drawn the lines Ce, &c. parallel to the joints MN, &c. meeting dQ in e, &c, if you take upon one of the legs of the right angle *g*c*a*, the parts $cm = \frac{\overline{KN}^2}{pp}$, &c. and upon the other the parts ce = Ce, &c. and you make always MN = me − cm, &c: the Curve paſſing thro' all the points H, M, &c. will be that required.

* Art. 2. For becauſe * $p=\frac{\overline{CA}^2}{CB}$, when the radius DM coincides with CA, then will KN ($=c$) $=$ CA; and $\frac{c^3}{pp}=\frac{\overline{CB}^2}{CA}=b$: and therefore DM will become $=b+AH$, and Ce $=$ Cd $=\sqrt{\overline{b+AH}^2-bb}$, and ſince cm $=\frac{\overline{KN}^3}{pp}=\frac{c^3}{pp}$, ce $=$ Ce, em will be $=\sqrt{\overline{Ce}^2+\frac{c^6}{p^4}}$, and em$-cm=NM=\sqrt{\overline{Ce}^2+\frac{c^6}{p^4}}-\frac{c^3}{pp}$, or NM $+\frac{c^3}{pp}=$ DM $=\sqrt{\overline{Ce}^2+\overline{DN}^2}$, becauſe DN $=\frac{c^3}{pp}$.

PROBLEM II.

Fig. 181. 305. *LET* BB *be a Piece of wood placed horizontally, and ſupported by the pieces* AB, AB, *which make a given angle* ABC *with the former; it is required to find the* Poſitions *of two other pieces* AC, AC, *given in length, ſuch that they ſhall ſupport the piece* BB *with the greateſt force poſſible.*

We ſuppoſe that the pieces AC are fixed in A and C in ſuch a manner as not to ſlip, and we neglect their gravity.

If BH be $=\frac{1}{2}$AC, and from the points A, H, the lines AG, HK, are drawn at right angles to BB: then if AC expreſſes the abſolute ſtrength of the piece AC, AG will expreſs the ſtrength with which it ſupports the piece BB: and therefore AG multiplied by the lever BC of the piece AC, ought to be a *Maximum.* Now if AC$=a$, HK$=n$, KB$=m$, AG$=x$; then will GC $=\sqrt{aa-xx}$, and becauſe of the ſimilar triangles HKB, AGB, we ſhall have HK : KB : : AG : GB $=\frac{m}{n}x$; and ſo BC $=\sqrt{aa-xx}-\frac{m}{n}x$, and AG×BC$=x\sqrt{aa-xx}-\frac{m}{n}xx$; whoſe Fluxion being ſuppoſed = 0, will give $\dot{x}\sqrt{aa-xx}-\frac{xx\dot{x}}{\sqrt{aa-xx}}-\frac{2m}{n}x\dot{x}=0$. Hence we get $x^4-aaxx+aann=0$, becauſe $(\overline{BH}^2=\overline{HK}^2+\overline{KB}^2)$ $4nn+4mm=aa$: and the ſquare root of this Equation will be $xx-\frac{1}{2}aa\pm am=0$; and ſo $x=\sqrt{\frac{1}{2}aa\mp am}$.

COROLLARY I.

306. IF we ſubſtitute for x its value $\sqrt{\frac{1}{2}aa\mp am}$ into the Equation $y=x\sqrt{aa-xx}-\frac{m}{n}xx$, we ſhall find $y=\frac{2ann-aam\pm 2amm}{2n}$. Now becauſe when $x=0$, y will be likewiſe $=0$; the firſt value $y=\frac{2ann-aam+2amm}{2n}$ will be* a *Maximum*, and the ſecond $y=\frac{2ann-aam-2amm}{2n}$ a *Minimum* if poſitive, or a *Maximum* if negative. For that Expreſſion cannot from poſitive become negative without becoming $=0$; and therefore the *Minimum* between thoſe *Maximums* is $=0$.

If the angle ABC inſtead of being obtuſe, were acute and the complement to this; we ſhall find $y=x\sqrt{aa-xx}+\frac{m}{n}xx$; whoſe fluxion being ſuppoſed $=0$ will give the ſame value for x as before; and ſubſtituting for x its value $\sqrt{\frac{1}{2}aa\mp am}$ into that Equation, we ſhall find $y=\frac{2ann+aam\mp 2amm}{2n}$: The firſt value $y=\frac{2ann+aam-2amm}{2n}$ will be a *Minimum*, and the ſecond $y=\frac{2ann+aam+2amm}{2n}$ a *Maximum*. Conſequently $(x=)$ $\sqrt{\frac{1}{2}aa-am}$ will be the Expreſſion of the ſine AG of the angle ACG required, when the angle ABC is obtuſe; and $(x=)$ $\sqrt{\frac{1}{2}aa+am}$, the Expreſſion of the ſine of the angle ACB, when the angle ABC is acute and equal to the Complement of ABC to two right ones.

---

* *Art.* 181.

COROLLARY II.

307. BEcauſe the ſum of the Squares of the Sines of any two angles, whoſe Sum is equal to a right angle, is always equal to the Square of the radius; the angle whoſe ſine is $\sqrt{\frac{1}{2}aa-am}$ will be the Complement of the angle whoſe ſine is $\sqrt{\frac{1}{2}aa+am}$, to a right one; ſince $\frac{1}{2}aa-am+\frac{1}{2}aa+am=aa$.

If the angle ABG be 60 degrees, then will BK $(=m)=\frac{1}{4}a$, and $AG=x=\sqrt{\frac{1}{2}aa-am}=\frac{1}{2}a$. Therefore the angle ACB will be 30 degrees; and conſequently if the angle ABC is 60 degrees, the angle ACB will be likewiſe 60 degrees.

But if ABC be a right angle, $m=KB$ will be equal to 0, and $AG=x=\sqrt{\frac{1}{2}aa}$, therefore the angle ACB will then be 45 degrees.

PROBLEM III.

FIG. 182. 308. L*ET* AG *be a Piece of wood of an indefinite length, and fixed in* A, *ſo as to make a given angle* GAD *with the horizontal line* AD: *it is required to find the* Poſition *of another piece* DE *given in length, ſuch that it ſhall ſupport the piece* AG *with the greateſt ſtrength poſſible.*

If $AC=\frac{1}{2}a$, the perpendicular CB to AD, $=n$, $AB=m$; and if from the point D the line $DF=x$ be drawn at right angles to AG: then if $DE=a$ expreſſes its ſtrength, when preſſed perpendicularly, DF will expreſs that with which it ſupports AG; therefore DF multiplied by the diſtance from the *Fulcrum* A, that is, by the lever AE, ought to be a *Maximum*. But becauſe of the ſimilar triangles ACB, ADF, we ſhall have CB : BA : : DF : FA $=\frac{m}{n}x$; and ſo $AE=\sqrt{aa-xx}+\frac{m}{n}x$ (becauſe $FE=\sqrt{aa-xx}$)

and AE × DF $= y = x\sqrt{aa - xx} + \frac{m}{n}xx$: whoſe Fluxion being ſuppoſed $= 0$, will give $x = \sqrt{\frac{1}{2}aa \pm am}$, which is the ſame Expreſſion as in the preceding Problem.

COROLLARY.

309. IF the angle ADE inſtead of being obtuſe was acute; then the point E will fall between the points A, F, and AE×DF $= y = -x\sqrt{aa - xx} + \frac{m}{n}xx$; whoſe Fluxion being ſuppoſed $= 0$, will give the ſame value for x as before. Therefore $x = \sqrt{\frac{1}{2}aa + am}$ will expreſs the Sine DF of the angle DEF in either Caſe: ſince that value of x will give a greater value for y, than any other. And becauſe $y = \frac{\pm 2ann + aam + 2amm}{2n}$; it is evident that the poſition of DE is more advantagious, when ADE is an obtuſe angle than when it is acute.

PROBLEM. IV.

310. LET ABCD *be an inclined plane, on which a cubic Box* LT *open at top, is drawn uniformly by a given Power* P *in a direction parallel to the plane* AC: *it is required to find the angle* BAK, *contained by that plane and the horizontal line* AK, *ſuch that the Power* P *ſhall draw out of a Reſervoir* R, *moſt Water that is poſſible in a given time.* FIG. 183.

If the height KB of the plane is given; then the greater the angle BAK is, the ſhorter will be the inclined plane or the way of the Box, and if EFNV is the level of the Water, then the greater the Priſm ELGHNV or only its altitude GF (becauſe the other dimenſions are given) is, the more Water the Box will contain. Therefore the product of the ſine of the angle A multiplied by the altitude GF of the priſm, ought to be a *Maximum.*

Now if from the center B with the radius BK, the circular arch Km be deſcribed, and the line mp be drawn perpendicular to KB; Bp will be the ſine of the angle Bmp = angle BAK. But if BK=Bm=a, Bp=x, EI=IG=1; mp will be=$\sqrt{aa-xx}$, and becauſe of the ſimilar triangles mpB, EIF, we have mp : pB :: EI : IF=$\frac{x}{\sqrt{aa-xx}}$; therefore FG=$1-\frac{x}{\sqrt{aa-xx}}$, and pB × FG=$x-\frac{xx}{\sqrt{aa-xx}}$: whoſe fluxion being ſuppoſed =0, will give $\dot{x}-\frac{2x\dot{x}}{\sqrt{aa-xx}}-\frac{x^3\dot{x}}{\overline{aa-xx}^{\frac{3}{2}}}=0$, hence we get $\overline{aa-xx}^{\frac{3}{2}}=2aax-x^3$. If a=10; then will x be nearly equal to $\sqrt{17}=4.1231$: and therefore, the angle whoſe radius and ſine are as 10 to 4.1231, will be equal to 24 degrees and 21 minutes =angle BAK.

* *Art.* 181. It is evident that the value found for x, * makes $y=x-\frac{xx}{\sqrt{aa-xx}}$ a *Maximum*. For x and y begin both together to increaſe from nothing, until x becomes =4.1231, then y decreaſes till x becomes = $\sqrt{\frac{1}{2}aa}$, when y is = 0 again, and then continues to increaſe negatively *ad infinitum*.

PROBLEM V.

FIG. 184. 311. LET AD *be the breadth of a River; it is required to find the* Poſition *of the Gates* AE, ED *of a Sluice, ſuch that they ſhall reſiſt againſt the Preſſure of the Water with the greateſt force poſſible.*

Upon AD as a diameter deſcribe the ſemi-circular Circumference ABD; then the radius BC drawn thro' the angle E, will be perpendicular to the line AD; and if from the interſection M of one of thoſe Gates produced, with the circumference, the Chord MD be drawn: then becauſe the reſiſtance of the Gates diminiſhes in the ſame proportion as the preſſure of the Water increaſes; and that preſſure is as the Surface on which it preſſes, if the height of the gates are given, AE will expreſs its reſiſtance. But the reſiſtance of the wood diminiſhes in proportion as its length increaſes; therefore $\overline{AE}^2$ expreſſes the whole reſiſtance of the Gate AE: and becauſe $\overline{AE}^2 : \overline{AC}^2 :: \overline{AD}^2 : \overline{AM}^2$, or $\frac{\overline{AC}^2}{\overline{AE}^2} = \frac{\overline{AM}^2}{\overline{AD}^2}$: the Square $\overline{AM}^2$, will expreſs the ſtrength of the Gate AE; ſince that Square increaſes in the ſame proportion as the ſquare $\overline{AE}^2$ (which expreſſes its reſiſtance) diminiſhes.

But the Gates preſs each other with more or leſs force, in proportion as the angle AED is greater or leſſer: Therefore if EF be drawn at right angles to AE, and FH perpendicular and FG parallel to the radius BC, and if FG expreſſes the reſiſtance of the Gate AE; FE will expreſs the reſiſtance of CB againſt the Gate AE in the direction EF; and becauſe that reſiſtance or force is equivalent to the forces FH and HE, the former perpendicular and the latter parallel to BC; it follows, that FH expreſſes the reſiſtance of BC againſt AE in a perpendicular direction. Therefore FH x $\overline{AM}^2$ ought to be a *Maximum*.

If AD $=a$, DM $=x$; then will $\overline{AM}^2 = aa - xx$; and becauſe of the ſimilar triangles ADM, AEC, FGE, EFH, we ſhall have AE : AC :: FG : EF and ſince both the lines AE, FG expreſs the reſiſtance of the gate AE againſt the water, FG will be $=$ AE and EF $=$ AC $=\frac{1}{2}a$: But AD : DM :: EF $(\frac{a}{2})$: FH $=\frac{1}{2}x$; therefore FH x $\overline{AM}^2 = \frac{1}{2}x \times \overline{aa-xx}$. Whoſe fluxion being ſuppoſed $=0$ will give $aa\dot{x} - 3xx\dot{x} = 0$ or $aa - 3xx = 0$, and $\sqrt{\frac{1}{3}aa} = x$.) Which ſhews that the angle DAM ought to be 35 degrees and 16 minutes.

It is manifeſt that the value found for x makes $y = \frac{aax - x^3}{2}$ * a *Maximum*. For x and y begin both together to increaſe from nothing

* *Art.* 181.

nothing until x becomes equal to $\sqrt{\frac{1}{3}aa}$, then y begins to decreaſe till x becomes $=a$, when y is $=0$ again.

PROBLEM VI.

312. *Having given any Engine moved by a Fluid as Water, Wind, Flame,* &c. *and ſerving to elevate any Weights: it is required to find the* Charge *of that Engine, ſo as to produce the greateſt Effect poſſible in a given time.* FIG. 185.

Let GFH be a trundle, fixed about the axis A of a wheel KCT moved by a Fluid in a direction QB; and let the ſpindles of the trundle, be taken by the cogs of a ſpur wheel RHM about the axis L of which, is fixed a Cylinder NIO, which Cylinder ſerves to wind up the cord NP ſuſtaining a weight P. Now if AB expreſſes the diſtance of the center of motion of the pallet B from the center of the axis A, and AH, HL, the diſtances of the point of Contact H of any of the ſpindles and cogs from the centers A, L; it is known in *Mechanics*, that if the weight *P*, be ſuch as to keep the Engine at reſt, the product of the radii LI, AH, of the Cylinder NIO and of the trundle GHF, multiplied by the Weight P, will be equal to the Product of the radii LH, AB, of the ſpur wheel HMR and of the great wheel CTK, multiplied by the force with which the fluid ſtrikes the pallet B. Therefore, if $AB=a$, $LH=b$, $AH=c$ $LI=d$, and B expreſſes the force of the fluid, we ſhall have in caſe of equilibrium $(B\times AB\times LH=P\times AH\times LI)$ $abB=cdP$.

If we ſuppoſe the weight P diminiſhed ſo that the Engine may move; then if the velocity of the fluid be v, and that of the center of motion of the pallet B, x; it is plain that the fluid will ſtrike the pallet B, only with the difference $v-x$ of thoſe velocities; and if we imagine a weight z to be applied inſtead of P, ſuch that it reſiſts againſt the force, when the fluid ſtrikes the pallet B with a velocity $v-x$; then becauſe the force of the fluid is as the ſquare of its velocity multiplied by the product of the pallet and the number of particles, which ſtrike at a time; and the number of particles as well as the Surface of the pallet B, are here given; the ſquare of the velocity $v-x$ will be the ſame with regard to the weight z, as the ſquare of the velocity v is with regard to the weight P; and as the levers are likewiſe the ſame in both caſes, we ſhall have $vv : \overline{v-x}^2 :: P : z$, or $zvv = P\times\overline{v-x}^2$: hence $v\sqrt{z} = \sqrt{P}\times\overline{v-x}$, and $x = \frac{v\sqrt{P}-v\sqrt{z}}{\sqrt{P}}$.

Again, the product ab of the radii AB, LH, will be to the product cd of the radii AH, LI; as the velocity x of the center of motion of the pallet B, is to the velocity of the weight z, $= \frac{cdx}{ab} = \frac{cdv}{ab}\times\frac{\sqrt{P}-\sqrt{z}}{\sqrt{P}}$. This velocity being multiplied by the weight z, will give $\frac{cdvz}{ab}\times\frac{\sqrt{P}-\sqrt{z}}{\sqrt{P}}$ for the exponent of the effect of the Engine. Conſequently if z be variable and the reſt of the quantities conſtant, and we make this exponent a *Maximum*; we ſhall have ſuch a value for the weight z, with which the Engine being charged will produce the greateſt effect poſſible. Therefore its fluxion being ſuppoſed $=0$, will give $\dot{z}\times\overline{\sqrt{P}-\sqrt{z}}-\frac{z\dot{z}}{2\sqrt{z}}=0$, or $2\sqrt{P}z-3z=0$, and $\frac{4}{9}P=z$. Which ſhews, that *if the Engine be firſt charged ſo as to remain juſt at reſt, and afterwards only with the* $\frac{4}{9}$*ths of that charge, it will produce the greateſt Effect poſſible.*

If z begins to increaſe from nothing, the Expreſſion $\frac{cdvz}{ab}\times\frac{\sqrt{P}-\sqrt{z}}{\sqrt{P}}$ begins likewiſe to increaſe from nothing, until z becomes $=\frac{4}{9}P$, then it decreaſes till z is $=P$, when that Expreſſion is nothing again. Therefore the value found for z, will* make $\frac{cdvz}{ab}\times\frac{\sqrt{P}-\sqrt{z}}{\sqrt{P}}$ a *Maximum*.

* Art. 181.

COROLLARY.

313. HEnce 1°. ſubſtituting for z $(=\frac{4}{9}\mathrm{P})$ its value into $\frac{cdvz}{ab} \times \frac{\sqrt{\mathrm{P}}-\sqrt{z}}{\sqrt{\mathrm{P}}}$, we ſhall have $\frac{4cdv\mathrm{P}}{27ab}$ or (becauſe $(\mathrm{P}=\frac{ab\mathrm{B}}{cd})=\frac{4v\mathrm{B}}{27}$ for the exponent of the Engine in its greateſt perfection. 2°. Subſtituting for z its value $\frac{4}{9}\mathrm{P}$ into $x=\frac{v\sqrt{\mathrm{P}}-v\sqrt{z}}{\sqrt{\mathrm{P}}}$, we ſhall have $x=\frac{1}{3}v$. Which ſhews, that *the velocity* (x) *of the center of motion of the pallet* B, *will be equal to the third part of the velocity* v *of the fluid, when the Engine is in its greateſt perfection.* 3°. Dividing the Exponent $\frac{4cdv\mathrm{P}}{27ab}$ of the effect of the Engine by the weight $\frac{4}{9}\mathrm{P}$, we ſhall have $\frac{cdv}{3ab}$ for the velocity of the weight z $(=\frac{4}{9}\mathrm{P})$.

PROBLEM VII.

FIG. 186. 314. LET *the line* CB *of a given thickneſs, be turned about the point* C *as a Center, by a weight* Q *tied to a ſtring* QEB, *which paſſes over a Pulley* E, *and is fixed to the extremity* B; *and let the perpendicular diſtance* CE *of the pulley* E *from the Center* C, *be*=CB; *it is required* 1°. *to determine the weight* Q, *ſo as to keep the line* CB *when placed in an horizontal Situation, in Equilibrium,* 2°. *to find ſuch a* Curve KLM, *that in whatſoever poſition the line* CB *may be, the weight* Q *placed in that Curve ſhall remain in equilibrium with that line,*

If CA be=AB, ſo that the point A be the Center of gravity of the line CB, in which we ſuppoſe to be united the whole weight of that line. Now it it is known in *Mechanics*, that in caſe of Equilibrium, the weight A multiplied by the perpendicular CA to its direction, will be equal to the weight Q multiplied by the perpendicular CX to its direction BE; therefore CA×A=CX×Q, that is, the weight Q, will be to the weight A, as the ſide (CA) of a Square, is to its diagonal (CX).

If K be the place of the weight Q, when the line CB is in an horizontal poſition; and L another place of the ſame weight, when in any other poſition as CH; draw L*a* perpendicular to CE, and from the Center of gravity m of the line CH, draw mp perpendicular to the horizontal line CB; then it is evident, that if the weight A multiplied by its perpendicular aſcent pm, be equal to the weight Q multiplied by its vertical deſcent K*a*, that is, if $pm \times A = Ka \times Q$; thoſe weights will be in equilibrium; and becauſe $CA : CX :: Q : A$, we ſhall have $pm \times CX = Ka \times CA$. But whilſt the point B deſcribes the circular quadrant BHIE, the Center of gravity A will deſcribe the quadrant Amno; therefore if the poſition CH of the line CB is given, the aſcent pm will be given, and conſequently the deſcent K*a* of the weight Q: and ſince the ſtring or chord BEQ or HEL is given, the line EL and conſequently L*a* are likewiſe given. And therefore the Curve KLM may be deſcribed.

LEMMA II.

315. THE Force *with which any Fluid ſtrikes a* Plane *or line* DN *in an oblique direction* KC, *will be to the force with which that Fluid would ſtrike the ſame in a perpendicular direction, as the* Square *of the* Sine *of the incident angle* KCD *is to the Radius.* FIG. 187.

About DN as a diameter deſcribe the circular circumference DKBNA, draw the diameter AB perpendicular and DP parallel to the direction KC of the Fluid; then if the radius KC expreſſes the Velocity with which the Fluid would ſtrike AB, the perpendicular KE (=PC) will expreſs the velocity with which the Fluid ſtrikes the plane DC; and ſince the number of particles which would ſtrike AB at a time, is to the number which ſtrikes DN, as AB to 2PC, or as AC to PC, and the force of the fluid is as the ſum of the products of all the particles each multiplied by its velocity: it follows, that the Square $\overline{PC}^2$ $(= \overline{KE}^2)$ of the ſine of the incident angle KCD, is to the Square $\overline{AC}^2$ of the radius, as the force with which the Fluid ſtrikes DN, is to the force with which it would ſtrike AB.

COROLLARY.

316. UNequal Planes DN, rs, unequally inclined to the direction of a fluid, are ſtruck by forces proportional to the products of thoſe planes each multiplied by the ſquare the of ſine of its incident angle. For let the plane rs be biſected by the center C, and from the interſection of that plane, produced if neceſſary, and the circumference the line dp be drawn perpendicular to AB, ſo that pC be the ſine of the incident angle KCd: then if P, Q, R, expreſs reſpectively the forces with which the planes DC, dC, rC, are ſtruck: we ſhall have * $\overline{PC}^2 : \overline{pC}^2 :: P : Q$; But unequal planes equally inclined, are ſtruck by forces proportional to the ſame planes. Therefore $dC : rC :: Q : R$; and multiplying orderly, $dC \times \overline{PC}^2$ or $DC \times \overline{PC}^2 :: rC \times \overline{pC}^2 :: P : R$.

* Art. 315.

PROBLEM. VIII.

Fig. 188. 317. *TO find the* Angle *which the plane of the Sails* E, F, *of a Windmill ought to make with the axis* AB, *ſo that a wind blowing uniformly in a direction* bd *parallel to the axis, ſhall make it turn with the greateſt ſwiftneſs poſſible.*

Let CBD be the angle, which the Sails ought to make with the axis AB, or with the direction of the wind, and ſuppoſe $\overline{BD}^2$ to expreſs the force with which the wind would ſtrike the Sails in a perpendicular direction; then the ſquare $\overline{DL}^2$ of the ſine of the angle of incidence DBC, will expreſs the force with which the wind impels the Sails in an oblique direction. But the force DL is equivalent to the forces LH and DH, the firſt perpendicular and the laſt parallel to the axis AB; and therefore LH will be the force with which the wind compels the Sails to turn.

Farther DL is to LH or BD to BL, as the force $\overline{DL}^2$ is to the force $\frac{BL\times\overline{DL}^2}{BD}$, with which the wind compels the Mill to turn, which ought to be a *Maximum*. Now if BD$=a$, BL $=x$; then will $\overline{DL}^2=aa-xx$; therefore $\frac{BL\times\overline{DL}^2}{BD}=\frac{aax-x^3}{a}$; whoſe fluxion being ſuppoſed $=0$, will give $aa\dot{x}-3xx\dot{x}=0$, or $aa=3xx$; and $x=\sqrt{\frac{1}{3}aa}$. Conſequently the angle BDL is 35 d. and 16 m. and the angle ſought DBL=54 d. 44 m.

PROBLEM IX.

318. L*ET* BC *be the Stern of a Ship moving forward in a direction* BC; *it is required to find the angle* CBA *or* LAB, *which the Rudder* AB *ought to make with the Stern, ſuch that the water ſhall compel the Ship to turn with the greateſt ſwiftneſs poſſible.* Fig: 189.

It is evident, that it is the ſame, whether the Ship moves forward in the direction BC, or the water ſtrikes the Rudder in the direction CB. Therefore if $\overline{AK}^2$ expreſſes the force of the water in a perpendicular direction, the ſquare of the ſine KD of the incident angle KAD, will expreſs the force in the oblique direction LA; and becauſe the force KD is equivalent to the forces DH, KH, the firſt perpendicular and the laſt parallel to the direction of the Ship; DK will be to DH or KA to AD, as the force $\overline{KD}^2$ is to the force $\frac{AD\times\overline{KD}^2}{KA}$ with which the water compels the Ship to turn. Therefore if AK$=a$, AD$=x$; then will $\overline{KD}^2=aa-xx$, and $\frac{AD\times\overline{KD}^2}{KA}=\frac{aax-x^3}{a}$: which being made a *Maximum*, that is, its fluxion being ſuppoſed $=0$, will give $aa\dot{x}-3xx\dot{x}=0$, or

$x=\sqrt{\frac{1}{3}aa}$. Conſequently the angle AKD will be 35 d. 16 m and the angle ſought KAD, =54 d. 44 m.

It is evident, that if the Rudder makes an angle (of 125 d. 16 m.) equal to the Complement of the angle 54 d. 44 m. the water will ſtrike the Rudder with the ſame force as before. The ſame thing may be ſaid with reſpect to the ſails of a Windmill.

PROBLEM X.

FIG. 190. 319. *TO find the* Expreſſion *of the* Reſiſtance *of Figures moving in a Fluid.*

Suppoſe a Fluid to ſtrike the Curve-line AMB in the direction LP, or the figure to move forward in the direction PL: then if in the tangent at M, you take $Mm=\dot{z}$ equal to the fluxion of the Curve, and you draw mr perpendicular to ML and mL to mM, * $mr=\dot{x}$ will expreſs the fluxion of the abſciſs CP, and $Mr=\dot{y}$ that of the Semi-ordinate PM. Now becauſe the force of each particle of the fluid in a perpendicular direction, is to the force in the oblique direction LM, as the radius LM is to the ſine Lm of the angle of incidence; or by reaſon of the ſimilar triangles, as Mm to mr. Therefore the force with which the particle M of the Curve is ſtruck, will be as Lm $=\frac{\dot{x}}{\dot{z}}$ (by ſuppoſing the abſolute force = 1.) But the force Lm is equivalent to the forces Lr, mr, the firſt parallel and the laſt perpendicular to the direction of the fluid. Whence if Lm : Lr or mM : mr :: $\frac{\dot{x}}{\dot{z}} : \frac{\dot{x}^2}{\dot{z}^2}=$ Lr; this fourth term $\frac{\dot{x}^2}{\dot{z}^2}$ will be as the reſiſtance of the particle M in the direction ML; and if Lm : rm or mM : Mr :: $\frac{\dot{x}}{\dot{z}} : \frac{\dot{y}\dot{x}}{\dot{z}^2}=$ mr, the fourth term $\frac{\dot{y}\dot{x}}{\dot{z}}$ will be as the reſiſtance of that particle in the direction mr or CB.

* Art. 164, 165.

* Art. 317.

If theſe reſiſtances are multiplied by the fluxion $mr=\dot{x}$, we ſhall have, rm $=\frac{\dot{y}\dot{x}^2}{\dot{z}^2}$, and Lr $=\frac{\dot{x}^3}{\dot{z}^2}$ for the fluxion of the reſiſtance of the arch AM. Therefore the fluents of which will give the Expreſſions of the reſiſtances required.

COR.

COROLLARY.

320. IF $\frac{c}{r}$ expreſſes the conſtant ratio of the radius to its circumference; then will $\frac{cy}{r}$, $\frac{cx}{r}$, be the circumferences, whoſe radii are PM and CP. Therefore the products of the circumference $\frac{cx}{r}$ multiplied by the fluxions of the reſiſtances, will give $\frac{cx\dot{y}\dot{x}^2}{r\dot{z}^2}$, $\frac{cx\dot{x}^3}{r\dot{z}^2}$ for the fluxions of the reſiſtances of the Surface generated by the rotation of the arch AM about the axis AC, *viz.* $\frac{cx\dot{y}\dot{x}^2}{r\dot{z}^2}$ the fluxion of the reſiſtance in the direction rm or BC, and $\frac{cx\dot{x}^3}{r\dot{z}^2}$ that of the reſiſtance in the direction LP or AC. And the circumference $\frac{cy}{r}$ multiplied by the fluxions $\frac{\dot{y}\dot{x}^2}{\dot{z}^2}$, $\frac{\dot{x}^3}{\dot{x}^2}$ will give $\frac{cy\dot{y}\dot{x}^2}{r\dot{z}^2}$, $\frac{cy\dot{x}^3}{r\dot{z}^2}$, for the fluxions of the reſiſtances of the Surface generated by the rotation of the ſame arch AM about the axis CB, in the directions rm or BC, and LP or AC.

EXAMPLE I.

321. LET the figure be a Triangle DAB, moving in a direction PL perpendicular to the baſe DB. If AC be drawn at right angles to DB, and if BP $= x$, BC $= a$, CA $= b$, BA $= d$; the ſimilar triangles BPM, BCA, will give BM $= z = \frac{dx}{a}$; and ſo $\dot{z} = \frac{d\dot{x}}{a}$. Therefore $\frac{\dot{x}^3}{\dot{z}^2} = \frac{aa\dot{x}}{dd}$ will be the fluxion of the reſiſtance in the direction PL; whoſe fluent will be $\frac{aax}{dd}$: FIG. 191.

and when BP=BC, ($x=a$), we ſhall have $\frac{a^3}{dd}=\frac{\overline{CB}^3}{\overline{BA}^2}$, for the reſiſtance of the ſide AB: and by the ſame reaſon $\frac{\overline{DC}^3}{\overline{DA}^2}$, will be the reſiſtance of the ſide DA. Therefore the Sum $\frac{\overline{CB}^3}{\overline{BA}^2}+\frac{\overline{DC}^3}{\overline{DA}^2}$ will expreſs the reſiſtance of the whole triangle in the direction PL.

If the triangle be equilateral, BC will be $=\text{DC}=\frac{1}{2}\text{AB}=\frac{1}{2}\text{DA}$, and $\frac{1}{4}$DB will expreſs its reſiſtance, which therefore is to the reſiſtance of the baſe DB, as unity is to 4.

EXAMPLE II.

322. LET the Conic-ſurface generated by the rotation of the line BA about the axis CA, move forward in the direction PL: then if BC$=a=r$, we ſhall have $\left(\frac{cx\dot{x}^3}{r\dot{z}^2}=\right)\frac{acx\dot{x}}{dd}$ for the fluxion of the reſiſtance; whoſe fluent will be $\frac{acxx}{2dd}$; and when $x=a$, we ſhall have $\frac{ca^3}{2dd}$ for the reſiſtance of the whole Conic-ſurface. Which therefore is to that of the circular baſe $(=\frac{ac}{2})$ deſcribed by CB, $\overline{CB}^2$ (aa) is to $\overline{AB}^2$ (dd).

EXAMPLE III.

Fig. 190. 323. LET the Curve AMB be a circular quadrant, CP$=u$, PM$=y$, CB=CA$=a$; then will $\dot{z}=\frac{a\dot{u}}{y}$, and $x=u$;

and if $r=a$, we ſhall have $\left(\frac{c\varkappa\dot{x}^3}{r\dot{z}\dot{z}}=\frac{cu\dot{u}yy}{a^3}\right)\frac{c}{a^3}\times\overline{aau\dot{u}-u^3\dot{u}}$ (becauſe $aa-uu=yy$); whoſe fluent $\frac{c}{a^3}\times\overline{\tfrac{1}{2}aauu-\tfrac{1}{4}u^4}$ will expreſs the reſiſtance in the direction PL of any part of the Surface generated by the rotation of the arch AM about the axis AC; and when $u=a$ (CP$=$CB), we ſhall have $\frac{1}{4}ac$ for the reſiſtance of the whole Surface: which therefore is to the reſiſtance of the circular baſe $(\frac{ac}{2})$ deſcribed by CB in that rotation, as unity is to 2.

EXAMPLE IV.

324. LET AMB be an elliptical quadrant, CP$=y$, PM$=u$, the firſt Semi-axis CA$=a$, the ſecond CB$=1$, then will* $aayy=aa-uu$ be the Equation of the Ellipſis; and becauſe $\varkappa=y$, we ſhall have $\frac{cy\dot{y}^3}{r\dot{z}^2}\left(=\frac{c\varkappa\dot{x}^3}{r\dot{z}^2}\right)$ for the reſiſtance of the Surface generated by the rotation of the arch AM about its firſt axis CA. But the Fluxion of the Equation $aayy=aa-uu$ will be $aay\dot{y}=-u\dot{u}$, or $\frac{a^4y^2\dot{y}^2}{uu}=\dot{u}^2=\frac{aay^2\dot{y}^2}{1-yy}$, and $\dot{u}+\dot{y}^2=\dot{y}^2+\frac{aay^2\dot{y}^2}{1-yy}=\dot{y}^2\times\frac{1-yy+aayy}{1-yy}=\dot{z}^2$, or ſuppoſing $aa-1=dd$, $\dot{y}^2\times\frac{1+ddyy}{1-yy}=\dot{z}$. Therefore the fluxion $\frac{cy\dot{y}^3}{r\dot{z}^2}$, will become $=\frac{cy\dot{y}-c\dot{y}y^3}{r+rddyy}=\frac{c}{rdd}\times$ into $\frac{\overline{dd+1}\times y\dot{y}}{1+ddyy}-y\dot{y}$. Conſequently if L$=$logarithm of $\sqrt{1+ddyy}$, the fluent of that fluxion will be $\frac{c}{rdd}\times$ into $\frac{dd+1}{dd}\text{L}-\frac{yy}{2}$: and when $y=1$, (CB$=$PM), then will $\sqrt{1+ddyy}$

* Art. 1

$=\sqrt{\overline{1+dd}}$, and $\frac{dd+1}{dd}$ L $-\frac{1}{2}$ × into $\frac{c}{rdd}$. Which therefore is to the reſiſtance of the circular baſe generated by CB in that rotation $(\frac{c}{2r}\overline{\mathrm{BC}}^2 =)\ \frac{c}{2r}$, as $\frac{2dd+2}{d^4}$ L $-\frac{1}{dd}$ to unity, or as $\overline{2dd+2}$ × L$-dd$ to d^4.

If CA be the ſecond axis; then becauſe $1>a$, $dd\ (=aa-1)$ will be negative; therefore $\frac{c}{rdd}$× into $\frac{1-dd}{dd}$ L$+\frac{yy}{2}$ will then be the Expreſſion required.

EXAMPLE V.

325. LET the Curve AMB be a Semi-parabola, A the vertex, CP$=y$, and the Semi-parameter of the axis AC, $=p$; * Art. 261. then will* $\dot{z}=\frac{\dot{y}}{p}\sqrt{\overline{pp+yy}}$; and becauſe CP$=x=y$; we ſhall have $\frac{cx\dot{x}^3}{r\dot{z}^2}=\frac{cppy\dot{y}}{rpp+ryy}$ for the Fluxion of the reſiſtance of the Surface, generated by the rotation of the arch AM about the axis AC, and $\frac{cpp}{r}$× into the L. of $\frac{\sqrt{\overline{pp+yy}}}{p}$ for the Expreſſion required.

PROBLEM XI.

326. OF *all the* Fruſtums *of a Cone, generated by the rotation of the trapezium* BCGH *about the axis* CG, *which have the ſame baſe* CB, *and the ſame height* CG; *to find that whoſe* Reſiſtance *is leaſt.*

Draw GH perpendicular and HT parallel to the axis CA: then if BT$=x$, BC$=a=r$, TH$=$CG$=b$; we ſhall have $\overline{\mathrm{CB}}^2-\overline{\mathrm{CT}}^2$ $=2ax-xx$, $\overline{\mathrm{BH}}^2=bb+xx$, and by the preceding Example

cxx

$\frac{cxx}{2a} \times \frac{2ax - xx}{bb + xx}$ will be the reſiſtance of the Surface generated by BH, and $\frac{c}{2a}\overline{GH}^2 = \frac{c}{2a} \times \overline{a-x}^2$ that of the Circle deſcribed by GH: theſe two expreſſions being added together, will give $\frac{c}{2a} \times \frac{aabb - 2abbx + bbxx + aaxx}{bb + xx}$, which ought to be a *Minimum.* Therefore its Fluxion being ſuppoſed equal to o, will give $\frac{2abbx^2\dot{x} + 2b^4x\dot{x} - 2ab^4\dot{x}}{\overline{bb+xx}^2} = 0$. Hence we get $xx + \frac{bb}{a}x = bb$, or $x = \frac{b}{2a}\sqrt{4aa+bb} - \frac{bb}{2a}$: Which gives the following Conſtruction,

Biſect CG in K, and make KA equal to KB; then will A be the vertex of the Cone. For $KB = KA = \frac{1}{2}\sqrt{4aa+bb}$, and $CA = \frac{1}{2}\sqrt{4aa+bb} + \frac{1}{2}b$, therefore AC : CB : : CG (HT) : TB; ſince the product of the means is equal to that of the extreams.

PROBLEM XII.

327. OF *all the* Surfaces *that may be deſcribed by the rotation of a Curve* MN (*about the axis* QP) *which paſſes thro'* Fig. 192. *two given points* M, N; *to find that whoſe* Reſiſtance *is leaſt, if moving in a perfect fluid, in the direction of the axis* QP.

Let the Fluxion of the Curve at M be *m*, at N, *n*, the Semi-ordinate PM equal to y, QN equal to z; then will * $\frac{cy\dot{y}^3}{rmm} + \frac{cz\dot{z}^3}{rnn}$ be the Sum of the Fluxions of the Reſiſtances at M and N; which ought to be a *Minimum.* But becauſe the Curve is to paſs thro' the points M, N, the lines PM (y), QN (z) are to be ta-

* Art. 320.

taken as conſtant; and as any two of the Fluxions may likewiſe be taken as conſtant: if m, n, are taken ſo, we ſhall have $\frac{y\ddot{y}\dot{y}^2}{mm} + \frac{z\ddot{z}\dot{z}^2}{nn} = 0$. Now if the Fluxions of the abſciſſes in P and Q are $\dot{x}$, $\dot{u}$; then will $mm = \dot{x}\dot{x} + \dot{y}\dot{y}$, and $nn = \dot{u}\dot{u} + \dot{z}\dot{z}$. The fluxion of the firſt Equation will be $-\dot{x}\ddot{x} = \dot{y}\ddot{y}$, and that of the laſt $-\dot{u}\ddot{u} = \dot{z}\ddot{z}$; and ſubſtituting for $\dot{y}\ddot{y}$ and $\dot{z}\ddot{z}$, their values into the Equation above, it will become $-\frac{y\dot{y}\dot{x}\ddot{x}}{mm} - \frac{z\dot{z}\dot{u}\ddot{u}}{nn} = 0$. But ſince PQ $(x+u)$ is given, $\dot{x} + \dot{u}$ will be equal to 0, or $\ddot{x} = -\ddot{u}$. Therefore $\frac{y\dot{y}\dot{x}}{mm} = \frac{z\dot{z}\dot{u}}{nn}$. Which ſhews that the Products $\frac{y\dot{y}\dot{x}}{mm}$, $\frac{z\dot{z}\dot{u}}{nn}$, ought always to be equal to each other, that is, $\frac{y\dot{y}\dot{x}}{mm}$ ought to be equal to a conſtant quantity (a): and conſequently $y\dot{y}\dot{x} = amm$ will be the Equation of the Curve required.

In order to free this Equation from the fluxionary Expreſſions, ſuppoſe $\dot{x} = \frac{\dot{z}\dot{y}}{a}$, and z to be a variable quantity; then becauſe $mm = \dot{y}^2 + \dot{x}^2$, the Equation $y\dot{y}\dot{x} = (amm =)\, a\dot{y}^2 + a\dot{x}^2$ will become $\frac{zy\dot{y}^2}{a} = a\dot{y}^2 + \frac{zz\dot{y}^2}{a}$, or $y = \frac{aa}{z} + z$: and taking the Fluxion of this laſt Equation, we ſhall have $\dot{y} = \dot{z} - \frac{\dot{z}aa}{zz}$. Therefore $\dot{x} = \left(\frac{z\dot{y}}{a} =\right) \frac{z\dot{z}}{a} - \frac{a\dot{z}}{z}$; whoſe fluent will be $x = \frac{zz}{2a} - \mathrm{L}az$, ſuppoſing $\mathrm{L}az$ to expreſs the logarithm of az.

If we ſuppoſe $y = \frac{aa}{z} + z = 0$, we ſhall find $aa + zz = 0$, or, $z = \sqrt{-aa}$; which ſhews that the Semi-ordinate PM cannot be equal to 0; and if we make $y = \frac{aa}{z} + z$ a *Minimum*; then will

—

$-\frac{aa\dot{z}}{zz}+\dot{z}=0$, or $aa=zz$, and $a=z$: and ſubſtituting this value of z, the Equation $y=\frac{aa}{z}+z$ will become $y=2a$; therefore the leſſer Semi-ordinate QN is $=2a$. And if we ſubſtitute for z its value a, the Equation $x=\frac{zz}{2a}-Laz$, will become $x=\frac{1}{2}a-Laa$. Conſequently if the abſciſſes begin at Q where the leſſer ordinate ſtands, we muſt ſubſtract the quantity $\frac{1}{2}a-Laa$, from the value of $x=\frac{zz}{2a}-Laz$, in order to have $x=\frac{zz}{2a}-\frac{1}{2}a-Laz+Laa$, or (becauſe $-Laz+Laa=-L\frac{z}{a}$), $x=\frac{zz}{2a}-\frac{1}{2}a-L\frac{z}{a}$ for the true Expreſſion of the abſciſs.

If the tabular Logarithm of $\frac{z}{a}$ be multiplied by the reciprocal Modulus* $R=2.302585$ &c., we ſhall have $R\times L\frac{z}{a}$ for the numerical expreſſion of that Logarithm, and $x=\frac{zz}{2a}-\frac{1}{2}a-R\times L\frac{z}{a}$.

N. B. Though the Curve MN does not meet the axis, yet it may approach very near it. For the Semi-ordinate QN ($=2a$) may be equal to any quantity as little as you pleaſe, provided it be a finite one; ſince the given quantity a, is taken at pleaſure.

If r be a third proportional, to QN ($2a$) and to z; and QN is ſuppoſed $=2$, ſo that a be $=1$; the Equation $x=\frac{zz}{2a}-\frac{1}{2}a-R\times L\frac{z}{a}$ will become $x=r-\frac{1}{4}QN-R\times Lz$; and if s, be a third proportional to z and to $\frac{1}{2}$ QN; the Equation $y=\frac{aa}{z}+z$, will be-

---

* Art. 243.

become $y=s+z$: Therefore if z be taken ſucceſſively equal to the Numbers 2, 3, 4, 5, &c: the quantities r and s, will be given; and multiplying ſucceſſively the Logarithms of theſe numbers by the *Modulus* R, the different values of $R \times Lz$ will be found likewiſe; and ſo there may be determined as many abciſſes as well as their correſpondent Semi-ordinates as you pleaſe.

PROBLEM. XIII.

Fig. 193.

328. TO *find a* Curve, *which ſhall paſs thro' two given points* M, n, *and ſuch, that a moving Body, whoſe velocity is every where as any power m of its perpendicular height from the beginning of its motion, ſhall paſs thro' it, in the ſhorteſt time poſſible.*

From the given points M, n, draw the lines M Q, n q, perpendicular to the horizontal line EB paſſing thro' the point E, from whence the the Body began to move. Now becauſe the ſpaces moved over, are as the times multiplied by the velocities acquired at the end of thoſe ſpaces, the times will be every where as the ſpaces divided by the velocities. Therefore if QM$=x$, qn$=v$, and the fluxion of the Curve at M, z, at n, n; then will $\frac{z}{x^m}$ expreſs the time in which the particle M of the Curve is moved over, and $\frac{n}{v^m}$ the time in which the particle n is moved over; and the ſum of theſe times will be $\frac{z}{x^m}+\frac{n}{v^m}$; which ought to be a *Minimum*. But becauſe the Curve is to paſs thro' the points M, n, the lines MQ(x) nq (v) are to be taken conſtant. Therefore the fluxion of $\frac{z}{x^m}+\frac{n}{v^m}$, being ſuppoſed equal to 0, will give $\frac{\dot z}{x^m}+\frac{\dot n}{v^m}=0$.

If the Fluxion of the abſciſs at Q be $\dot y$, that at q, $\dot u$; then will $zz=\dot x^2+\dot y^2$, $nn=\dot v^2+\dot u^2$; and the Fluxions of theſe Equations (by taking $\dot x$, $\dot v$ as conſtant) will be $z\dot z=\dot y\ddot y$, $n\dot n=\ddot u\dot u$, or $\dot z=\frac{\dot y\ddot y}{z}$ and $\dot n=\frac{\dot u\ddot u}{n}$; and ſubſtituting for $\dot z$, $\dot n$, their values into $\frac{\dot z}{x^m}$

+

$+\frac{\dot{n}}{v^m}=0$, it will become $\frac{\dot{y}\ddot{y}}{zx^m}+\frac{\dot{u}\ddot{u}}{nv^m}=0$. But ſince Qq $(y+u)$ is given, we ſhall have $\dot{y}+\dot{u}=0$, or $\ddot{y}=-\ddot{u}$: Conſequently $\frac{\dot{y}}{zx^m}=\frac{\dot{u}}{nv^m}$. Which ſhe ws that the products $\frac{\dot{y}}{zx^m}$ ought always to be equal to each other, that is, $\frac{\dot{y}}{zx^m}$, ouhgt to be equal to a conſtant quantity a^{-m}: therefore $\frac{\dot{y}}{zx^m}=a^{-m}$ or $\dot{y}a^m = zx^m$ will be the Equation of the Curve required.

If the line MK be drawn perpendicular to the tangent at M; then will MQ (x) : MK (c) : : $\dot{y} : z$, or $z=\frac{c\dot{y}}{x}$; and ſubſtituting this value of z, into $\dot{y}a^m=zx^m$, it will become $a^m = cx^{m-1}$.

COROLLARY.

329. IF $m=1$, that is, if the velocity is every where as the perpendicular height from the beginning of the motion; then will $a^m = cx^{m-1}$ become $a=c$. Which ſhews, that the perpendicular MK is always conſtant; and therefore the Curve EMn will be a circular Arch, whoſe Center is ſomewhere in the horizontal line EQ. But if $m=\frac{1}{2}$, that is, if the velocity is always as the ſquare root of the height MQ, that is, the velocity which a Body acquires by its own gravity; then will $a^{\frac{1}{2}}=cx^{-\frac{1}{2}}$ or $\sqrt{ax}=c=$ MK; which is the Equation of the common Cycloid. For if upon the axis BA$=a$ of that Curve, there be deſcribed a Semicircular circumference BNA, and from any of its points M, the line MP be drawn parallel to EB, interſecting the circumference in N and meeting BA in P: then the property of the Cycloid is ſuch, that the Chord BN is parallel and equal to the perpendicular MK: and by the property of the Circle AB×PB $(ax)=\overline{\text{BN}}^2=\overline{\text{MK}}^2$, or $\sqrt{ax}=$ MK. Therefore, &c.

PROB.

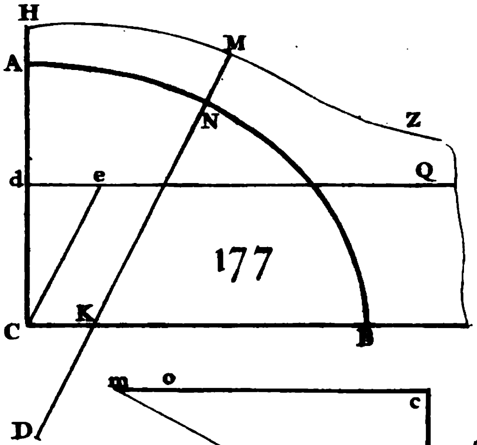
PROBLEM XIV.

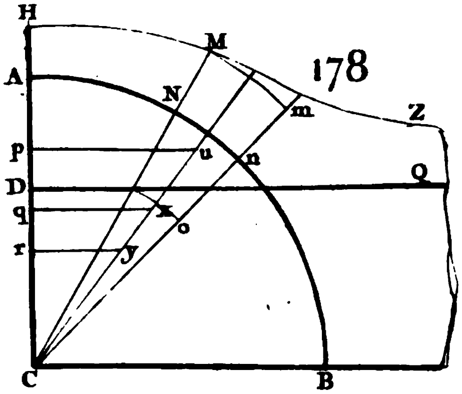
Fig. 194. 330. *LET a Body* Q *be fixed in* B, *to an inflexible line* DB *void of gravity; it is required to find the* Diſtance *of the center of* Oſcillation *of another Body* P *fixed to the ſame line, ſuch that the Pendulum ſhall perform its vibrations with the greateſt ſwiftneſs poſſible.*

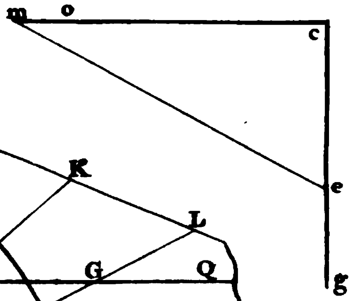
Suppoſe DB $(=a)$, DA $(=x)$, to expreſs the diſtances of the centers of Oſcillation of the Bodies Q (c), P (d) from the point of Suſpenſion D: then if n expreſſes the diſtance of their common Center of Oſcillation C, we ſhall have * $n = \frac{aad + xxc}{ad + xc}$. Now it is evident, that the ſhorter DC $(=n)$ is, the greater will be the celerity of the Pendulum; therefore $\frac{aad + xxc}{ad + xc}$, ought to be a *Minimum*; whence its fluxion being made equal to o, will give $\frac{2acdx\dot{x} + ccxx\dot{x} - aacd\dot{x}}{\overline{ad + xc}^2} = 0$, hence we get $xx + \frac{2ad}{c}x = \frac{aad}{c}$. Therefore $x = -\frac{ad}{c} + \frac{a}{c}\sqrt{dc + dd}$. And ſubſtituting for x this value into $n = \frac{aad + xxc}{ad + xc}$, we ſhall have $n = -\frac{2ad}{c} + \frac{2a}{c}\sqrt{dc + dd}$ for the diſtance of the center of Oſcillation C from the point of Suſpenſion D.

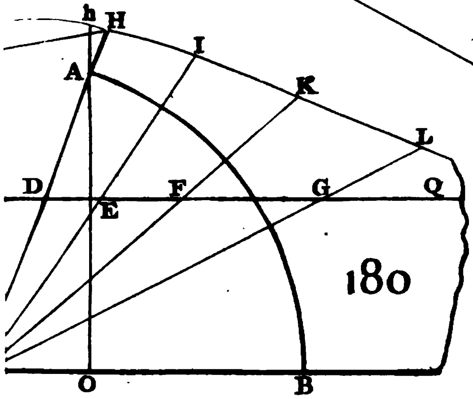
If in the Equation $n = \frac{aad + xxc}{ad + cx}$, x be ſuppoſed equal to o, it will become $n = a$; and if x be infinite, we ſhall have $n = x$, that is, n becomes likewiſe infinite, therefore that expreſſion contains* a *Minimum* and no *Maximum*. Art. 181.

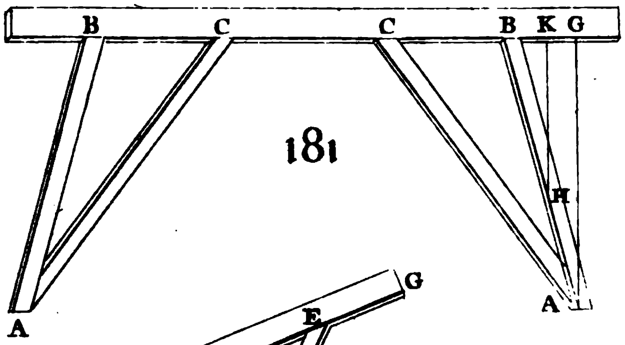
PRO-

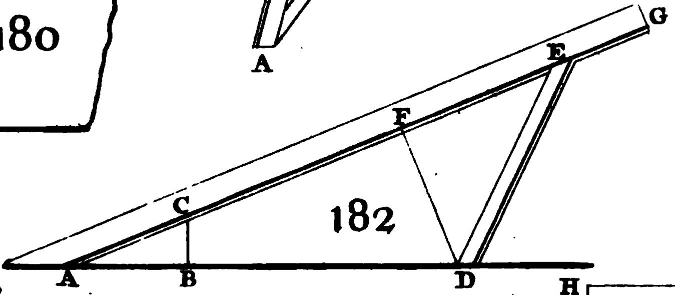

177

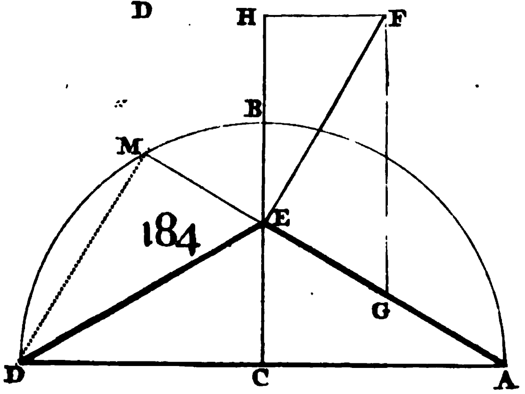

178

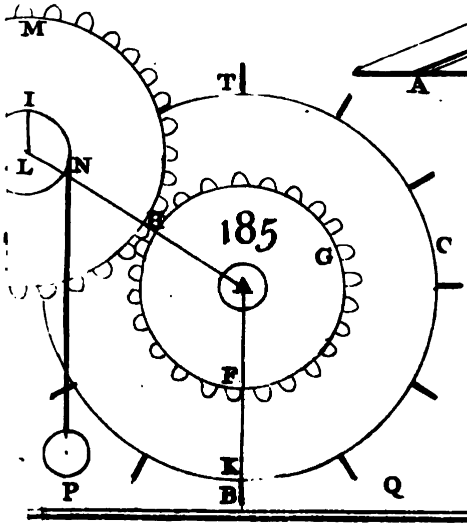



180


181


182


184


185

PROBLEM XV.

331. *THE diſtance of the center of Oſcillation* (n) *of a Sphere from the point of Suſpenſion being given, as well as the diameter of the Sphere* ($2a$); *to find the* Diſtance (d) *of the Sphere from the point of Suſpenſion.* FIG. 170.

Becauſe * $n = \frac{5dd+10ad+7aa}{5a+5d}$, we ſhall have $5an+5nd = 5dd+10ad+7aa$, or $dd+2ad-nd = an-\frac{7}{5}aa$, and adding $\frac{\overline{2a-n}^{2}}{4}$ to both ſides, it will be $dd+2ad-nd+\frac{\overline{2a-n}^{2}}{4} = \frac{1}{4}nn-\frac{2}{5}aa$, whoſe ſquare root will be $d+a-\frac{1}{2}n = \sqrt{\frac{1}{4}nn-\frac{2}{5}aa}$. Therefore $d = \frac{1}{2}n - a \pm \sqrt{\frac{1}{4}nn-\frac{2}{5}aa}$.

* Art. 394.

PROBLEM XVI.

332. *LET the common center of Oſcillation* C *of the Bodies* P, Q, *as well as that of the Body* Q *be given; to find the* Diſtance DA *of the Body* P *from the point of Suſpenſion* D. FIG. 194.

The ſame thing being ſuppoſed as in the Article 330; we ſhall have $n = \frac{aad+cxx}{ad+cx}$, or $\frac{and-aad}{c} = xx-nx$; and adding $\frac{nn}{4}$ to both ſides, then will $\frac{and-aad}{c}+\frac{nn}{4} = xx-nx+\frac{nn}{4}$; whoſe Square root will give $\frac{1}{2}n \pm \sqrt{\frac{and-aad}{c}+\frac{nn}{4}} = x$. Therefore, ſince the diſtance of the center of Oſcillation from the point of Suſpenſion of that Body is given, the diſtance required may be found by the help of the preceding Problem.

PROBLEM XVII.

Fig. 195. 333. T*O find the* higheſt *and the* loweſt Point *of the Helice of* Archimede's *Screw, the angle* AKB *made by the horizontal line* AK, *and by the Cylinder* ABDE *on which the Helice is formed, being given.*

Let AMBDYE be half the Cylinder, AmMB and DYE, the Semicircular baſes, CQ the axis, and ArRV the path of the Screw: and ſuppoſe R to be one of the points required; then the Sections of the plane paſſing thro' that point, parallel to the axis CQ and perpendicular to the plane ABDE, with the baſe AMB, the plane ABDE, the horizontal plane ASK and the convex Surface of the Cylinder, will form the rectangle PMST. Now if $AC=CB=a$, $CP=x$, $BV=d$, $BK=b$; AP will be equal to $a+x$, and becauſe of the ſimilar triangles APT, ABK, we ſhall have $AB:BK::AP:PT=MS=\frac{ab+bx}{2a}$; and by the nature of the *Helice*, the Semi-circumference AmMB (c) is to the height BV (d) of the Screw, as any arch AM (z) is to the correſpondent height $MR=\frac{dz}{c}$ of the ſame. Therefore $MS-MR=RS=\frac{ab+bx}{2a}-\frac{dz}{c}$, which ought to be a *Minimum*; whence its fluxion being made equal to o, will give $\frac{b\dot{x}}{2a}-\frac{d\dot{z}}{c}=0$, or $bc\dot{x}=2ad\dot{z}$. But by the property of the Circle, the fluxion $\dot{z}$ of the arch AmM is $=\frac{a\dot{x}}{\sqrt{aa-xx}}$: therefore $bc\dot{x}=(2ad\dot{z})\frac{2aad\dot{x}}{\sqrt{aa-xx}}$ or $\sqrt{aa-xx}=\frac{2aad}{bc}$, and $aa-\frac{4a^4dd}{bbcc}=xx$. Hence we get $CP=x=\pm\frac{a}{bc}\sqrt{bbcc-4aadd}$. Conſequently if $CP=Cp=\frac{a}{bc}\sqrt{bbcc-4aadd}$, and the lines PM, Pm are drawn perpenicular to AB, and from their Interſections M, m, with the Semi-circumference, the lines MS, ms, are drawn parallel to the axis CQ; they will interſect the *Helice* in the loweſt point R and in the higheſt r.

N. B. If this Screw be made uſe of in drawing up Water to any height, it is plain that the angle KAB ought to be ſuch, that $bbcc > 4aadd$.

For, ſince when $bc = 2ad$, the two points R, r, will coincide with the horizontal plane A s S K; therefore the level of the water will coincide with that of the Screw in the points R, r, united into one.

COROLLARY.

334. SUppoſe a Power N to be applied to the Semi-circumference AMB in a direction of the tangent to the circular baſe at B, and it be required to find the proportion of that power and the weight of the Water. Draw thro' the lower point R, the line RL perpendicular to the horizontal plane ASR, and in the ſame plane, the line FR perpendicular to MR, and from the axis CQ let fall GH perpendicular to FR produced: then if RL (p) expreſſes the weight of the Water, and LF is perpendicular to FR, the line FR will expreſs the power of the Water in the direction FH or BA. But becauſe FH is in a plane parallel to the plane ABK, GH will be equal to $TS = PM = \frac{2aad}{bc}$ (ſince $\overline{CB}^2 - \overline{CP}^2 = \overline{PM}^2$): and by reaſon of the ſimilar triangles ABK, LFR, we ſhall have AK (r) : BK (b) : : RL (p) : FR $= \frac{bp}{r}$: and ſo the product of GH or PM multiplied by FR, $= \frac{2aadp}{rc}$ ought to be equal to BC×N, in caſe of Equilibrium; therefore $rcN = 2adp$.

But if BX be drawn parallel and DX perpendicular to the horizontal line AK, the ſimilar triangles ABK, DXB, will give AK (r) : AB ($2a$) : : BD (m) : DX (n), or $r = \frac{2am}{n}$. Subſtituting for r its value into $rcN=2adp$, it will become $mcN=ndp$. Laſtly, by the property of the *Helice*, the Semi-circumference AMB (c), is to BV (d), as the Semi-circumference c multiplied by the number of turns (s) of the *Helice* before it arrives at D, is to the length BD of the Cylinder, that is, $d=\frac{m}{s}$; therefore $mcN=(ndp)\frac{nmp}{s}$. Conſequently $cs:n::p:N$; that is, the Power (N) is to the Weight (p), as the height of the Cylinder ($DX=n$) is to the path of the *Helice* or Way paſſed over by the Water.

PROBLEM XVIII.

Fig. 196. 335. *IF a Body falls freely thro' a vertical line* BA *and afterwards paſſes over a Curve line* AM; *it is required to find in each place, the relation of the* Times, Velocities *and* Spaces *moved over.*

Let the tangent MT at M expreſs the Fluxion $\dot{z}$ of the Curve; and the ſubtangent TP, and ſemi-ordinate PM, the Fluxions $\dot{x}$, $\dot{y}$, of the abſciſs AP and the Semi-ordinate PM; then if v, u, expreſs the velocity and the time from the beginning of the fall; we ſhall have, the product of the time multiplied by the velocity equal to the Space moved over; or the product of the velocity at multiplied by M the Fluxion of the time equal to the Fluxion of the Curve, that is $\dot{u}v=\dot{z}$, or $\dot{u}v=c\dot{z}$, ſuppoſing c equal to 1. Whence two of theſe unknown quantities being given the third will be given likewiſe.

EXAMPLE I.

336. *IF the velocity v is every where as any Power of the height* BT; *and it be required to find the* Nature *of the Curve* AM *ſuch, that the Body ſhall paſs over equal* Space *in equal* Times.

If AB$=a$, we ſhall have $v=\overline{a+x}^{m}$, and $u=a+x$, or $\dot{u}=\dot{x}$, therefore $\dot{u}\,v=c\dot{z}$, will become $\dot{x}\times\overline{a+x}^{m}=c\dot{z}$, or $\dot{x}\times\overline{a+x}^{m}=c^{m}\dot{z}$, becauſe $c=1$; whoſe Fluent will give $\frac{1}{m+1}\times\overline{a+x}^{m+1}=c^{m}z$ for the general Equation of all Curves, thro' which the Body paſſes equal vertical Spaces in equal Times, with a velocity every where as $\overline{a+x}^{m}$.

If the velocity be that which a Body acquires by its own gravity, that is, if $m=\frac{1}{2}$; we ſhall have $\dot{x}\sqrt{a+x}=\dot{z}\sqrt{c}=\sqrt{c\dot{x}^{2}+c\dot{y}^{2}}$; the ſquare of which $a\dot{x}^{2}+x\dot{x}^{2}=c\dot{x}^{2}+c\dot{y}^{2}$, gives (by ſuppoſing $c=a$) $x\dot{x}^{2}=a\dot{y}^{2}$, or $\dot{x}\sqrt{x}=\dot{y}\sqrt{a}$; whoſe fluent will be $\frac{2}{3}x^{\frac{3}{2}}=y\sqrt{a}$, or $x^{3}=\frac{9}{4}ayy$. Which ſhews that the Curve AM is a Semi-cubical Parabola, A its vertex and $\frac{9}{4}$ AB its parameter.

EXAMPLE II.

337. LET the Curve AM be the common Parabola, A its vertex and $ay=xx$, the Equation; and let it be required to find the times, ſuch that the velocities ſhall be every where as the ſquare root of the vertical height BP.

Becauſe $v=\sqrt{a+x}$, and the Fluxion of $ay=xx$, gives $\dot{y}=\frac{2x\dot{x}}{a}$; we ſhall have $\dot{z}=\sqrt{\dot{x}^{2}+\dot{y}^{2}}=\frac{\dot{x}}{a}\sqrt{aa+4xx}$, and ($v\dot{u}=c\dot{z}$) $\dot{u}\sqrt{a+x}=\frac{c\dot{x}}{a}\sqrt{aa+4xx}$, or $\dot{u}=\frac{c\dot{x}\sqrt{aa+4xx}}{a\sqrt{a+x}}$: the Fluent of which will give the time required.

If the times are as the ſpaces moved over, that is, if $a+x=u$, or $\dot{x}=\dot{u}$, and the velocity be required, the reſt being the ſame as before; we ſhall have $v\dot{x}=\frac{c\dot{x}}{a}\sqrt{aa+4xx}$, or $v=\frac{c}{a}\sqrt{aa+4xx}$, for the velocity in any place M, ſo that a Body moving in a Parabola paſſes over equal vertical ſpaces in equal times.

PROBLEM XIX.

Fig. 197. 338. *LET a Body moving in a Curve* AM *be always attracted towards a given point* C, *placed in the vertical line paſſing thro' the higheſt or loweſt point* A *of the Curve; it is required to find in each place, the relation between the* Times, Velocities *and* Spaces *moved over.*

From the Center C, with the radii CM, CA, deſcribe the circular arches MP, AN; meeting the lines CA, CM, produced if neceſſary, in P and N: then if the part Mr of the tangent at M to the Curve AM expreſſes the fluxion of the Curve, and rn be drawn at right-angles to Mn, a tangent to the circular arch MP; rn will be the Fluxion of CM, and Mn that of the circular arch PM: and retaining the ſame Expreſſions for the velocities and the times as in the preceding Problem, we ſhall have as before $v\dot{u}=c\dot{z}$. But if the arch $AN=y$, $CA=a$, $AP=x$; then will CP or $CM=a-x$, and CN : CM :: as the fluxion $\dot{y}$ of the arch AN is to the fluxion $Mn=\frac{a\dot{y}-x\dot{y}}{a}$ of the arch MP: therefore $\dot{z}^2=\dot{x}^2+\frac{\overline{a-x}^2}{aa}\dot{y}^2$, or $\dot{z}=\frac{1}{a}\sqrt{aa\dot{x}^2+\overline{a-x}^2\times\dot{y}^2}$. Conſequently any two of theſe unknown quantities being given the other will will be given alſo.

EXAMPLE.

339. *LET the Body be attracted towards the Center* C, *by a force which is every where as any Power m of the diſtance* CM, *of its place from the Center* C; *and let it be required to find the arch* AM *ſuch, that the Body ſhall paſs over equal vertical Spaces in equal Times.*

If the line CX be drawn perpendicular to r M produced, XM will expreſs the velocity of the Body at M : and becauſe of the ſimilar triangles M n r, CXM; we ſhall have Mr : rn : : CM $\overline{(a-x)}^m$: MX $= \frac{\dot{x}}{\dot{z}} \times \overline{a-x}^m = v$; and ſince $x=u$, or $\dot{x}=\dot{u}$, by ſuppoſition, we ſhall have $(v\dot{u}=c\dot{z})\frac{\dot{x}^2}{\dot{z}} \times \overline{a-x}^m = c\dot{z}$, or $\dot{x}^2 \times \overline{a-x}^m = (c^m \dot{z}^2) = \frac{c^m aa\dot{x}^2 + c^m \times \overline{a-x}^2 \times \dot{y}^2}{aa}$, and $aa\dot{x}^2 \times \overline{\overline{a-x}^m - c^m} = c^m \dot{y}^2 \times \overline{a-x}^2$; Whence we get $a\dot{x}\sqrt{\overline{a-x}^m - c^m} = c^{\frac{1}{2}m}\dot{y} \times \overline{a-x}$, for the general Equation required.

If the Body be attracted towards the Center C, every where by a force proportional to the Diſtance CM of its place from that Center; we ſhall have $m=1$, and $a\dot{x}\sqrt{ac-cx-cc} = c\dot{y} \times \overline{a-x}$, for the Equation required.

PROBLEM. XX.

340. *SUppoſe the Extention of a Spring when wound up on a Cylinder, to be every where as its length: it is required to find a* Curve DMV, *which revolving about the axis* CX *ſhall form ſuch a Surface, that if a Chain fixed to one end of the Spring and the other in* D, *and is rouled on that Surface; the ſpring will have every where an equal force to unroul that* Chain. FIG: 198.

Let V be the point of the Surface where the end of the chain when rouled up is, take in the perpendiculars DK, VX to the axis CX produced, the lines KE, XL, in the proportion of the extentions of the parts of the Spring anſwering to the points D and V of the Chain, and draw the line LE ſo as to meet the axis ſomewhere in C : draw likewiſe a Semi ordinate MP meeting LE in N. Now becauſe PN repreſents the ſtrength of the ſpring anſwering to the Point M of the Chain, PN multiplied by the lever PM ought always to be the ſame, that is, that product ought to be equal to a conſtant quantity *ab*. Therefore PN x PM$=ab$. But if CK$=a$, KP$=x$, EK$=b$; the ſimilar triangles

CKE, CPN, will give CK : KE : : CP : PN $= \frac{ab+bx}{a}$: Conſequently $aa = ay + xy$ will be the Equation of the Curve required, which will be an Hyperbola, C the Center and CX one of the aſymptotes.

COROLLARY.

341. Whatever the Law of the Extention of the Spring will be, the Curve DMV may be found provided that Law is known. For Example, ſuppoſe the Extention of the Spring to be every where as any power m of its length; then will $y \times \frac{\overline{ab+bx}^m}{a^m} = ab^m$, or $y \times \overline{a+x}^m = a^{m+1}$. But if PN$=\sqrt{2ax+xx}$, the Equation of the Curve required will be $aa = y\sqrt{2ax+xx}$; and ſo on in any other Caſe.

PROBLEM XXI.

FIG. 199. 342. TO *find the* Nature *of the Curve which a perfect flexible line* DAK, *fixed by its ends in a plane, will make if preſſed or drawn every where by Powers, whoſe ratio's to each other are given, either all in a direction perpendicular to* DK, *or every where perpendicular to the Curve.*

Caſe I. Draw O T at right-angles to the middle B of DK, interſecting the Curve in A, and from any point of the Curve, draw the Tangent MT as well as the Semi-ordinate MP to *AB*, and finiſh the rectangle MHTP. Now becauſe of the Equilibrium; the power in the direction MH or PT, will be to the powers in the directions TM, TH, of the tangents at M and A, as MH is to TM and TH; or (if the line O*g* perpendicular to the tangent TM, interſects DK in E), as the line BE to the lines * OE, OB, which are perpendicular to their directions. Likewiſe if the perpendicular O h to the tangent at m, interſects DK

* *Art.* 202.

DK in F; the power applied to the point m in a direction perpendicular to DK, will be to the powers in the directions of the tangents TM, mh, as EF is to EO and FO. And becauſe this happens always; it follows that the power applied to any particle M, is as the fluxion of the correſpondent line BE.

If AP$=x$, PM$=y$, AM$=z$, BO$=a$, BE$=u$; the ſimilar triangles BOE, PMT, will give PM $(\dot y)$: PT $(\dot x)$: : BO (a) : BE $= u = \frac{a\dot x}{\dot y}$; whoſe fluxion is $\dot u = \frac{a\dot y\ddot x - a\dot x\ddot y}{\dot y^2}$. But $\dot z^2 = \dot y^2 + \dot x^2$, by taking $\dot z$ conſtant, its Fluxion will give $-\frac{\dot y\ddot y}{\dot x} = \ddot x$; ſubſtituting for $\ddot x$ its value into the value of $\dot u$, we ſhall have $\dot u = \frac{a\dot z^2\ddot y}{\dot x\dot y^2}$, becauſe $\dot z^2 = \dot y^2 + \dot x$.

Caſe II. Draw HR perpendicular to TM; and finiſh the rectangle RMSH : then will HM : SM or TM : TH, as the power $\dot u$ in the direction HM, to the power SM $= \frac{\dot u\dot y}{\dot z}$ in the direction SM. But becauſe the directions of the powers SM, which are every where perpendicular to the Curve, are not parallel to each other, there will be a part of their actions deſtroyed : if therefore SM be reduced into two, the one SV perpendicular and the other VM parallel to AB; VM will expreſs the efficacy of the power SM, that is, if TM : TH : : SM : VM $= \frac{\dot y^2\dot u}{\dot z^2} = \frac{a\ddot y}{\dot x}$, this fourth term $\frac{a\ddot y}{\dot x}$ will be the expreſſion of the power M, in a perpendicular direction to the Curve.

EXAMPLE I.

343. T*O find the* Nature *of the Curve which a perfect flexible line will make, being preſſed by the* Atmoſphere *of the Air.*

From the Nature of Fluids, the preſſure will be every where perpendicular to the Curve, and as the product of the Surface preſſed multiplied by the altitude of the Fluid; and becauſe the difference of the altitudes of the Air which preſs any two points A, M of the Curve is ſo ſmall, that they may be taken as equal; the Force with which each particle M of the line is preſſed, will be as its Fluxion $\dot{z}$ in that point: whence $\dot{z} = \frac{a\ddot{y}}{\dot{x}}$, or $\dot{x}\dot{z} = a\ddot{y}$: and ſince $\dot{z}$ was ſuppoſed conſtant whilſt $\dot{x}$, $\dot{y}$ flow; the fluent of this Equation will be $x\dot{z} = a\dot{y}$: whoſe ſquare will be $(xx\dot{z}^2 =)$ $x^2\dot{y}^2 + x^2\dot{x}^2 = aa\dot{y}^2$, or $xx\dot{x}^2 = \dot{y}^2 \times \overline{aa - xx}$, whence $\frac{x\dot{x}}{\sqrt{aa-xx}} = \dot{y}$; whoſe Fluent $\sqrt{aa - xx} = y$ will give $aa - xx = yy$. Which ſhews that the Curve is a Circle in this Caſe.

EXAMPLE II.

344. *To find the* Nature *of the Curve which a perfect flexible line will make, being preſſed or filled up with any Fluid whatever.*

By the Nature of Fluids, if PB $= x$, $\dot{z}x$ will expreſs the Force with which the particle M is preſſed. Therefore * $\dot{z}x = \frac{a\ddot{y}}{\dot{x}}$, or
* Art. 342.
$2\dot{z}x\dot{x} = aa\ddot{y}$, by multiplying $a\ddot{y}$ by $\frac{a}{2}$ in order to make the Equation homogeneous; whoſe fluent will be $\dot{z}xx = aa\dot{y}$, the ſquare of this Equation is $(\dot{z}^2x^4 =)$ $x^4\dot{y}^2 + x^4\dot{x}^2 = a^4\dot{y}^2$, or $x^4\dot{x}^2 = \dot{y}^2 \times \overline{a^4 - x^4}$. Whence we get $xx\dot{x} = \dot{y}\sqrt{a^4 - x^4}$ for the Equation required.

EXAMPLE III.

345. *To find the* Nature *of the Curve which a perfect flexible Chain will make, being fixed with its Ends in a vertical Plane.*

Becauſe all the particles are ſuppoſed equal and of the ſame weight, the force with which the particle M is drawn vertically, will as

$\dot z$. Therefore * $\dot z = \frac{a\dot z^2 \ddot y}{\dot x \dot y^2}$, or $\dot x = \frac{a\dot z \ddot y}{\dot y^2}$: whoſe Fluent will be * Art. 342. $x = \frac{a\dot z}{\dot y}$, becauſe $\dot z$ is ſuppoſed conſtant: the ſquare of this Equation will give $\dot y^2 x^2 = (aa\dot z^2 =)\ aa\dot y^2 + aa\dot x^2$. Whence $\dot y^2 \times \overline{xx - aa} = aa\dot x^2$, or $\dot y\sqrt{xx - aa} = a\dot x$, will be the Equation required.

Becauſe * $u = z = \frac{a\dot x}{\dot y}$, if we ſubſtitute for $\frac{a\dot x}{\dot y}$ its value $\sqrt{xx - aa}$, * Art. 342. we ſhall have $z = \sqrt{xx - aa}$ for the Expreſſion of any arch AM. But when AM $= z = 0$; then will $\sqrt{xx - aa} = 0$, or $x = a$; therefore if the abſciſſes begin at the vertex A, we muſt ſubſtitute $a + x$ inſtead of x; Conſequently $\dot y\sqrt{2ax + xx} = a\dot x$ will then be the Equation of the Curve, and $z = \sqrt{2ax + xx}$.

EXAMPLE IV.

246. TO *find the* Nature *of the Curve which a perfect flexible line will make, being preſſed by the* Wind *blowing uniformly in a perpendicular direction to* DK.

Becauſe of the uniform motion of the Wind, the force with which any particle M is ſtruck or preſſed, will be as $\dot z$; therefore $\dot z = \frac{a\dot z^2 \ddot y}{\dot x \dot y^2}$. Which is the ſame Equation as that in the preceding Example.

EXAMPLE V.

347. TO *find the* Nature *of the Curve which a perfect flexible line will make, being preſſed in each point by a force which is as any power m of its diſtance from* DK.

We ſhall have $\dot z x^m = \frac{a\dot z^2 \ddot y}{x\dot y^2}$, or $\dot x x^m = \frac{\dot z \ddot y a^{m+1}}{\dot y^2}$ (ſuppoſing

$a = 1$), whoſe fluent will be $\frac{1}{1+m}x^{n} = \frac{\dot z}{\dot y}a^{m+1}$. Therefore if the exponent m is given, the Equation may be found in the ſame manner as in the preceding Examples.

COROLLARY I.

348. IT is evident, that whatever the Law of the Powers, which preſs or draw a perfect flexible line according to any directions, may be; the Nature of the Curve may be found. For the ſaid directions may be changed into others either perpendicucular to DK, or every where perpendicular to the Curve.

COROLLARY II.

349. IT is likewiſe evident, that if the Nature of the Curve be given, the law of the Powers, which preſs a perfect flexible line, may be found, ſo that it ſhall make the Curve required. This we ſhall illuſtrate by the following Examples.

EXAMPLE I.

350. *TO determine the Law of the powers, which preſſing each particle of a perfect flexible line in ſuch a manner, that the Curve ſhall be a* Circle.

If AB = BD = a, BP = x, PM = y; then will $yy = aa - xx$ be the Equation of the Circle; whoſe Fluxion will give $\dot y = -\frac{x\dot x}{y}$. Therefore * $u = \left(\frac{a\dot x}{\dot y} =\right) - \frac{ay}{x}$; the Fluxion of which will be $\dot u = \frac{ay\dot x - ax\dot y}{xx}$: and if p expreſſes the force which preſſes the particle M, we ſhall have $\dot u = p\dot z = \frac{ay\dot x - ax\dot y}{xx}$.

* Art. 342.

But by the property of the Circle $\dot{z} = \frac{a\dot{x}}{y}$; whence ſubſtituting for $\dot{z}, -\dot{y}$, their values, we ſhall find $\frac{ap\dot{x}}{y} = \frac{axx\dot{x} + ayy\dot{x}}{yxx}$, or $p = \frac{aa}{xx}$, becauſe $aa = xx + yy$.

EXAMPLE II.

551. *TO determine the* Law *of the Weights, which preſſing each particle of a perfect flexible line in ſuch a manner, that the Curve ſhall be a* Parobola.

Let $2ax = yy$ be the Equation; whoſe Fluxion will give $\dot{x} = \frac{y\dot{y}}{a}$. Therefore $u = \left(\frac{a\dot{x}}{\dot{y}}\right)y$; and $\dot{u} = \dot{y}$: and if p expreſſes the weight which preſſes the particle M, we ſhall have $p\dot{z} = \dot{y}$. But by the property of the Parabola, * $\dot{z} = \frac{\dot{y}}{a}\sqrt{aa + yy}$, conſequently $(p\dot{z} =) \frac{p\dot{y}}{a}\sqrt{aa + yy} = \dot{y}$, or $p = \frac{a}{\sqrt{aa + yy}}$. * Art. 261.

N. B. If each particle of the Curve made by a perfect flexible line, be conſidered as a Vouſſoir of a Vault or Arch, whoſe interior or exterior Curve be that formed by the line; it is evident, that they will remain in Equilibrium with each other, provided their Weights are in the ſame proportion as the preſſures by which means the Curve is formed, obſerving only that the directions of their Centers of gravity ought to paſs thro' that particle, of which preſſure they are made proportional.

DEFINITION.

The Curve-line EMAmF deſcribed by a given point M, of a circumference of a Circle GMTV moving over a right line EBF, is called a *Cycloid*, or if moving over a Circular earch EBF, an *Epicycloid*: the line EBF is called the *baſe*, the right line AB perpendicular to the middle of the baſe, the Axis, the point A the *Vertex*, and GMTV the generating Circle. FIG. 200, 201.

COROLLARY I.

352. HEnce it follows, 1°. that the baſe EBF is equal to the circumference of the generating Circle GMTV, and the part EG between the point of contact G and the beginning E, equal to the arch GM, and therefore BG, equal to the arch MT complement of the arch GM to the Semi-circumference. 2°. The chord GM will be perpendicular to the Tangent MT at M. For the difference of the circular velocities of the points G, M; that is, the velocity with which the Curve is deſcribed in M, will be * as the radius GM, and in the direction perpendicular to GM as well as in the direction of the * tangent MT.

* Art. 189.
* Art. 164.

3°. If about the axis AB as a diameter, the circular circumference BNAn be deſcribed, interſecting the line MP parallel to EB in N; the chords BN, GM will be parallel and equal. Therefore BG = MN = arch MT or NA, and PM = PN + arch NA.

Fig. 202.

4°. If $AB=a$, $AP=x$; then will $AN=\sqrt{ax}$; and becauſe AN, TM are parallel, we ſhall have, AP : AN or AN : AB :: $\dot{x} : \frac{a\dot{x}}{\sqrt{ax}}$ = to the Fluxion of the arch AM; whoſe fluent will be $2\sqrt{ax}=2AN$. Therefore the arch AM is double the correſpondent chord AN, and the Semi-cycloid AME double the diameter AB.

5°. If from the center O of the circular baſe EBF, and with the radius OM, be deſcribed the circular arch MN; then the arches GM, BN, as well as the chords MG, NB will be equal. For ſince the lines OT, OA paſs thro' the Centers c, C, of the equal Circles GMTV, BNAn, and OB = OG; ON = OM; the arches BN, GM, and conſequently the chords BN, GM will be equal.

Fig. 201.

COROLLARY. II.

353. IF MT expreſſes the Fluxion of the Curve EM, and the line Tr be drawn at right angles to OM produced, Mr will expreſs the Fluxion of OM; and if from the points B, N, the lines BX, NP are drawn at right-angles to ON and OA;

the triangles BXN, MrT will be ſimilar, ſince the angle MTr is equal to the angle GMO=angle BNO. Therefore, if OB=b, AB=a, AP=x, PN=y, ON=OM=z, we ſhall have ON : NP : : OB : BX $= \frac{by}{z}$, and AN ($\sqrt{ax}$) : AB : : PN : BN$=\frac{ay}{\sqrt{ax}}$; Laſtly, BX : BN : : Mr : MT$=\frac{az\dot{z}}{b\sqrt{ax}}$. But ($\overline{ON}^2=\overline{OP}^2+\overline{PN}^2$) $zz=aa+bb-2bx-ax$; whoſe Fluxion is $z\dot{z}=-\dot{x}\times\overline{b+\frac{1}{2}a}$: and ſo $-$MT $= (\frac{az\dot{z}}{b\sqrt{ax}} = \frac{a+2b}{2b}\times\frac{a\dot{x}}{\sqrt{ax}}$, the Fluent of which, $=\frac{a+2b}{b}\times\sqrt{ax}$ will be the expreſſion of the arch AM, and $\overline{a+2b}\times\frac{a}{b}$ that of half the Epicycloid AME.

COROLLARY III.

354. IF Ka expreſſes the Fluxion of the chord Km = Bn, the line ab drawn at right angles to Km, meeting the baſe BF in b *fig.* 200, or the tangent Kb to the baſe at K *fig.* 201, will be the circular velocity of the point K with regard to the radius KD, ſuppoſing the point D to be in the evolute FDI. Now becauſe of the right-angled ſimilar triangles AnB, Kab, we have An : Bn : : Ka : ab = to the * Fluxion of An, that is, ab$=\frac{a\dot{x}}{2\sqrt{ax}}$. But if mt expreſſes the fluxion of the arch Am, we ſhall have * tm : ab : : mD : DK, or by diviſion tm$-a$b : tm : : mK : mD. Conſequently ſince * tm$=\frac{a\dot{x}}{\sqrt{ax}}$ *fig.* 200, we ſhall have mD=2mK=2Bn: and becauſe * tm $=\frac{a+b}{b}\times\frac{a\dot{x}}{2\sqrt{ax}}$ *fig.* 201, we ſhall find mD $=\frac{a+2b}{a+b}$ Bn.

FIG. 200.
FIG. 201.

* *Art.* 197.
* *Art.* 194.
* *Art.* 352. Nº. 4.
* *Art.* 353.

COR-

COROLLARY IV.

355. IT is evident, that $\overline{tm+ab} \times \frac{1}{2}mK$ will be the Fluxion of the Space FmK. And becauſe $tm=2ab$, in *fig.* 200; and $mK=Bn$; that Fluxion will be $=\frac{1}{2}Bn\times ab$, that is, three times the Fluxion of the Segment BxnB; ſince ab is equal to the Fluxion An, or equal to the circular velocity of the point n with reſpect to the radius Bn. Therefore the ſpace FmK is triple the Segment BxnB; and the Area of the ſemi-Cycloid FMAB triple the Semi-generating Circle.

* Art. 353. But ſince* $tm=\frac{a+2b}{b}\times\frac{a\dot{x}}{2\sqrt{ax}}$ in *fig.* 201, and $ab=\frac{a\dot{x}}{2\sqrt{ax}}$: therefore $\overline{tm+ab}\times\frac{1}{2}mK=\frac{a+3b}{b}\times\frac{a\dot{x}}{2\sqrt{ax}}$ Bn, or becauſe $Bn\times\frac{a\dot{x}}{2\sqrt{ax}}$ is the Fluxion of the Segment BxnB; we ſhall have $\frac{a+3b}{b}\times BxnB$ for the Expreſſion of the ſpace FmK; and $\frac{a+3b}{b}\times BxnA$ for that of the Semi-Epicycloid FmABKF.

COROLLARY V.

Fig. 200. 356. THE evolute FDI of the Cycloid FmA, will be another Cycloid equal to the former. For let FH be = AB and perpendicular to FB, and let the line DQ parallel to BF, interſect the Semi-circumference whoſe diameter is HF, in L, and HF in Q: then becauſe $mK=KD=Bn$, FQ will be $=BP$, $QL=Pn$, and $LD=FK=$ arch $Bn=$ arch FL. Therefore, ſince this happens every where, FDI will be another Cycloid equal to FmA; F the vertex, HF the axis, and HI half the baſe.

Fig. 201. If Kk be $=\frac{ab}{a+b}$, the point D of the circumference KDk, whoſe diameter is Kk, will deſcribe in the rotation of that Circle over the circular arch fkI whoſe radius is Ok, another Epicycloid IDF, which will be the Evolute of the former FmA. For becauſe when the point m coincides with the vertex A, mD becomes $=AI=\frac{aa+2ab}{a+b}$, and $OI=Ok=\frac{bb}{a+b}$; therefore tK

(a)

$(a) : Kk(\frac{ab}{a+b} :: KO(b) : kO = \frac{bb}{a+b} ::$ arch KF : arch kf; and ſince the arches FK, mK, are always equal as well as the arches KB, tm; the arches Dk, kI, and conſequently the arches KD, fk, will be likewiſe equal.

COROLLARY VI.

357. IF the generating Circle GMTV, inſtead of moving without the Circle EGBF, did move within the ſame; the point M would deſcribe another ſort of Epicycloid, whoſe property will be the ſame as thoſe of the former EMAmF, except AB (a) which before was poſitive, will be here negative. Therefore by changing the Signs of the odd powers of a, all what has been ſaid with reſpect to this will be true with reſpect to the other.

PROBLEM XXII.

358. *IF a Body only compelled by its gravity, moves over a Cycloid* EMA, *whoſe baſe* EB *is placed horizontally, and its vertex* A *downwards; it is required to find the* Time, *in which the Body moves over any arch* MA.

Fig. 193.

Becauſe $\sqrt{AP}$ expreſſes the velocity which the Body acquires by moving over the arch MA, and $\sqrt{AP}$ is as AN or as half the arch MA; it follows that the velocity is as the ſpace moved over; and therefore the time is given and equal to the time in which the Body would move over the Semi-cycloid EMA.

COROLLARY.

359. IF $BP = x$, $AB = a$, 2AN or the arch AM will be $= 2\sqrt{aa - ax}$, and $\sqrt{x}$ will expreſs the velocity at M. Therefore the Fluxion $\frac{-a\dot{x}}{\sqrt{aa - ax}}$ of the ſpace EM divided by the velocity $\sqrt{x}$, will

give $\frac{-\dot{x}\sqrt{a}}{\sqrt{ax-xx}}$ for the Fluxion of the time. But $\frac{a\dot{x}}{\sqrt{ax-xx}}$ is the Fluxion of the arch BN: conſequently $\frac{BN}{\sqrt{a}}$ will expreſs the time in which the arch EM is moved over, and $\frac{BNA}{\sqrt{a}}$, that in which the Semi-cycloid is moved over. And ſince $\frac{a}{\sqrt{a}} = \sqrt{a}$ expreſſes the time of a falling Body thro' BA: *the time in which a Body moves over the Semi-cycloid* EMA *will be to the time in which that Body would fall thro'* AB, *as the Semi-circumference* BAN *is to its diameter* BA.

PROBLEM. XXIII.

FIG. 201. 360. *TO find the* Time *in which a Body would move over any arch* MA *of the Epicycloid* EMA, *with a centripetal force every where proportional to its diſtance from the center* O.

Draw OY perpendicular to TM produced; then YM will expreſs the velocity of the Body at M; and becauſe of the ſimilar triangles TG : TM :: GO : MY; and ſince TG, GO, are given, the velocity MY will be as the Fluxion TM of the Curve AM; and as this happens always, the velocity with which any arch MA is moved over, will be as that arch; and therefore the time will be given and equal to the time in which half the Epicycloid EMA is moved over.

COROLLARY.

361. IF BP $=x$, then will $\overline{ON}^2 = \overline{OM}^2 = zz = bb + 2bx + ax$, AP $= a - x$, and the Fluxion of the Curve at M will be* $\overline{b+\frac{1}{2}a} \times \frac{a\dot{x}}{b\sqrt{aa-ax}}$: and drawing Yz perpendicular to OM, MY will be to Mz, as the Fluxion of the Curve is to the Fluxion of OM; and becauſe of the ſimilar triangles OM : MY :: MY : Mz, or

* Art. 353

or $\text{OM}\times\text{M}z=z\dot{z}=b\dot{x}+\frac{1}{2}a\dot{x}=\overline{\text{MY}}^{2}$, and the ſquare root of the fluent $bx+\frac{1}{2}ax$ will expreſs the velocity at M, or becauſe $b+\frac{1}{2}a$ is given, that velocity will be as $\sqrt{x}$. Therefore the Fluxion of the Curve $\overline{b+\frac{1}{2}a}\times\frac{a\dot{x}}{b\sqrt{aa-ax}}$ divided by the velocity $\sqrt{x}$, will give $\frac{b+\frac{1}{2}a}{b}\times\frac{\dot{x}\sqrt{a}}{\sqrt{ax-xx}}$ for the Fluxion of the time: But $\frac{a\dot{x}}{\sqrt{ax-xx}}$ is the fluxion of the arch BN: therefore $\frac{b+\frac{1}{2}a}{b}\times\frac{\text{BN}}{\sqrt{a}}$ will be the Expreſſion of the time in which the Body moves over the arch EM, and $\frac{b+\frac{1}{2}a}{b}\times\frac{\text{BNA}}{\sqrt{a}}$ that of the time in which the Body moves over half the Epicycloid, which therefore is to the time in which that Body would fall thro' AB as $\frac{b+\frac{1}{2}a}{b}$ BNA to AB.

PROBLEM. XXIV.

362. TO *find the* Nature *of the Curve* EAF, *paſſing thro' two given points* M, N, *ſuch that a Body with any Centripetal force ſhall paſs over it in the ſhorteſt time poſſible.* FIG. 202.

From the Center of forces C, with the radii CM, CN, deſcribe the circular arches MP, NQ, meeting the vertical line CB in P, Q; draw CT perpendicular to the tangent MT at M, and the line TR perpendicular to CM: then if MT ($\dot{z}$) expreſſes the Fluxion of the Curve in M, MR will expreſs the Fluxion of the line CM.

Now if CM expreſſes the centripetal force at M, MT will expreſs the force or velocity in the direction MT; ſince the force CM is equivalent to the force CT with which the Body is drawn towards the Center C, and to that MT with which it moves in the Curve: and becauſe of the ſimilar triangles CM : MT :: MT : MR, or $\text{CM}\times\text{MR}=\overline{\text{MT}}^{2}$. Therefore if the perpendicular PH to CP expreſſes the Centripetal force at M, and the Curve GHIK is the locus of all the points H; the ſquare root of the ſpace PBGH will expreſs the velocity of the Body at M acquired by moving thro' EM. Likewiſe, if the perpendicular QI to AB meets the Curve HK in I; $\sqrt{QBGI}$ will expreſs the velocity at N. Whence if v be the fluxion of the arch EN, and $c = \sqrt{PBGH}$, $d = \sqrt{QBGI}$; we ſhall have $\frac{z}{c} + \frac{v}{d}$ for the Sum of the fluxions of the times in M, N; which ought to be a *Minimum*. but becauſe the points M, N or P, Q, are given, the ſpaces PBGH, QBGI, are likewiſe given, and ſo $\frac{\dot{z}}{c} + \frac{\dot{v}}{d} = 0$; and ſince the points M, N, are given, the fluxions of CM, CN, are conſtant; therefore if y, u, expreſs the fluxions of the arches MP, NQ; we ſhall have $\dot{z}z = y\dot{y}$, $v\dot{v} = u\dot{u}$, or $\dot{z} = \frac{y\dot{y}}{z}$, $\dot{v} = \frac{u\dot{u}}{v}$; whence $\frac{\dot{z}}{c} + \frac{\dot{v}}{d} = 0$, will become $\frac{y\dot{y}}{zc} + \frac{u\dot{u}}{vd} = 0$. But Becauſe the angle MCN is given, the Sum $y + u$ of the fluxions of the arches MP, QN will be $= 0$, or $\dot{y} = -\dot{u}$. Conſequently $\left(\frac{y\dot{y}}{zc} + \frac{u\dot{u}}{vd} = 0\right) \frac{y}{zc} = \frac{u}{vd}$: which ſhews that the quantity $\frac{y}{zc}$ ought to be every where the ſame; that is, equal to a conſtant quantity $\frac{1}{a}$. Whence $\frac{y}{zc} = \frac{1}{a}$, or $ay = cz$.

COROLLARY.

363. HEnce, if from the loweſt point A of the Curve, the line AK be drawn at right-angles to CB meeting the Curve GHI in K; then will $ABGK = aa$. For when the point M coincides with the point A, z will be $= y$; and therefore $a = \sqrt{ABGK}$. Likewiſe, if the fluxion of CM be $= \dot{x}$, $y\sqrt{APHK} = c\dot{x} = \dot{x}\sqrt{PBGH}$: Since $aay^2 = (cczz)\, ccyy + cc\dot{x}\dot{x}$, or $\overline{aa - cc} \times yy = cc\dot{x}\dot{x}$, or $y\sqrt{aa - cc} = c\dot{x}$. Therefore $\dot{x}\sqrt{ABGK}$ or $a\dot{x} = z\sqrt{APHK}$.

EXAMPLE I.

364. IF the Center C removes at an infinite diſtance, ſo that the lines CM, CN, become parallel and equal to CA; then the Semi-ordinates PH, AK, become conſtant, ſuppoſe $=a$; and if AP$=x$, the ſpace APHK will be $=ax$. Therefore $a\dot{x}$ ($=z\sqrt{\text{APHK}}$)$=z\sqrt{ax}$. Which is the Equation of the Cycloid EMA. For if AB$=a$, AP$=x$; then will AN$=\sqrt{ax}$, FIG. 193. and * $z:\dot{x}::$ (AN : AP : :) AB (a) : AN ($\sqrt{ax}$.) * *Art.* 352. N°. 4.

EXAMPLE II.

365. IF the centripetal force be always as any power $m-1$ of the diſtance of the Body from the Center C: then if CM$=$CP$=x$, CB$=b$, CA$=a$; we ſhall have $\dot{x}x^{m-1}$ for the Fluxion of the ſpace PBGH, and $\frac{1}{m}x^m=$ PBGH. But when CP$=$CB($x=b$), that ſpace ought to be $=0$; therefore $\frac{1}{m}b^m-\frac{1}{m}x^m=$ PBGH: and by the ſame reaſon APHK$=\frac{1}{m}x^m-\frac{1}{m}a^m$. Conſequently $y\sqrt{x^m-a^m}=\dot{x}\sqrt{b^m-x^m}$. Now if the line CM produced meets the arch BE in e, and if the arch Be$=v$; then will CB (b) : CP (x) :: $\dot{v}:y=\frac{\dot{v}x}{b}$; ſubſtituting for y its value, we ſhall have $\frac{\dot{v}x}{b}\sqrt{x^m-a^m}=\dot{x}\sqrt{b^m-x^m}$, or $\dot{v}=\frac{b\dot{x}}{x}\sqrt{\frac{b^m-x^m}{x^m-a^m}}$.

If $z=\sqrt{\frac{b^m-x^m}{x^m-a^m}}$, then will $zzx^m-zza^m=b^m-x^m$, or $x=\dfrac{\overline{b^m+a^mzz}|^{\frac{1}{m}}}{\overline{1+zz}|^{\frac{1}{m}}}$; whoſe fluxion will be $\dot{x}=\frac{2}{m}\times$ into

$a^m \dot{z}z \times \frac{\overline{b^m + a^m zz}|^{\frac{1}{m}-1}}{\overline{1+zz}|^{\frac{1}{m}}} - z\dot{z} \times \frac{\overline{b^m + a^m zz}|^{\frac{1}{m}}}{\overline{1+zz}|^{\frac{1}{m}+1}}$. Now the firſt ſide of this Equation being divided by $\frac{x}{z}$, and the ſecond by its equal $\frac{1}{z} \times \frac{\overline{b^m + a^m zz}|^{\frac{1}{m}}}{\overline{1+zz}|^{\frac{1}{m}}}$ we ſhall have $\frac{z\dot{x}}{x} = \frac{2}{m} \times$ into $\frac{a^m zz\dot{z}}{b^n + a^m zz} - \frac{zz\dot{z}}{1+zz}$. But $\frac{a^m zz\dot{z}}{b^m + a^m zz}$ is $= \dot{z} - \frac{b^m \dot{z}}{b^m + a^m zz}$, and $\frac{zz\dot{z}}{1+zz} = \dot{z} - \frac{\dot{z}}{1+zz}$. Therefore $\dot{v} = \frac{z\dot{x}b}{x} = \frac{2b}{m} \times$ into $\frac{b^m \dot{z}}{b^m + a^m zz} - \frac{\dot{z}}{1+zz} =$ to the Fluxion propoſed. Conſequently if D expreſſes the circular arch whoſe radius and tangent are as $\sqrt{\frac{b^m}{a^m}}$ to z, and d the arch whoſe radius and tangent are as unity to z; we ſhall have the arch Be $= v = \frac{2}{m} b$D $- \frac{2}{m} b$d: and ſo the arch Be being given as likewiſe CM, the point M of the Curve required will be given alſo.

If the centripetal force were reciprocally as any power $m+1$, of the diſtance of the Body from the center C; then will $\frac{1}{x^m} - \frac{1}{b^m} = \frac{b^m - x^m}{b^m x^m} =$ PBGH, and $\frac{1}{a^m} - \frac{1}{x^m} = \frac{x^m - a^m}{a^m x^m} =$ APHK. Therefore $\dot{v} = \frac{a^m b\dot{x}}{b^m x} \sqrt{\frac{b^m - x^m}{x^m - a^m}}$. Conſequently if $z = \sqrt{\frac{b^m - x^m}{x^m - a^m}}$, and D expreſſes the arch whoſe radius and tangent are as $\sqrt{\frac{b^m}{a^m}}$ to z; and d the arch whoſe radius and tangent are as unity to z; then will $v = \frac{2a^m b}{m b^m}$D $- \frac{2a^m b}{m b^m}$d: Whence the point M will be found in the ſame manner as before.

PROB.

PROBLEM XXV.

366. TO *find the* Nature *of the Curve which a Ray of Light deſcribes paſſing thro' a Medium, whoſe denſity at equal diſtances from a given Center, is equal, and increaſes by approaching the Center according to any Law.*

From the given Center C deſcribe the circular arches PM, QN, meeting the Curve ſought in M, N, and the vertical line CA in P, Q: draw the lines AG, PH, QI, perpendicular to CA, and ſo as to be to each other as the denſities in thoſe places. Now if the equal parts MT, NV, of the tangents at M, N, expreſs the Fluxions of the Curve in thoſe points, and the lines TR, VS are drawn at right-angles to CM, CN; then MR, NS will expreſs the Fluxions of the lines CM, CN; and becauſe if MT, NV, expreſs the velocities with which the Ray paſſes thro' the particles M, N; MR, NS, will expreſs the velocities with which that Ray approaches the Center C. But it is known by Experiments, that the velocity of light increaſes in proportion as the denſity increaſes. Therefore NS : MR :: QI : PH, that is, if NV $=$TM$=\dot{z}$, NS$=\dot{x}$, QI$=v$, PH$=c$, we ſhall have MR$=\frac{c\dot{x}}{v}$: and if the incident angle in any given place M, is given, the ratio of TM to MR will be likewiſe given. Conſequently if TM : MR : : $m : n : : \dot{z} : \frac{c\dot{x}}{v}$, we ſhall have $\dot{z}v=\frac{cm}{n}\dot{x}$ for the Equation of the Curve required. FIG. 203.

Becauſe the denſity c (PH) in the given place P is given, if $\frac{cm}{n}=a$; then will $\dot{z}v=a\dot{x}$ be the Equation.

EXAMPLE. I.

367. LET the Center C be removed at an infinite diſtance, ſo that the lines CM, CN, become parallel and equal, and let the denſity be every where as the power m of the Depth

of the medium, then will $\dot{z}x^m = a^m\dot{x}$ be the Equation. If m
FIG. 200. $= \frac{1}{2}$, we ſhall have $\dot{z}\sqrt{x} = \dot{x}\sqrt{a}$, or $\dot{z}\sqrt{ax} = a\dot{x}$, which is the Equation of the Cycloid AME. For if AP$=x$, AB$=a$; then will $\dot{z} : \dot{x} ::$ (AN : AP ::) AB (a) : AN ($\sqrt{ax}$).

EXAMPLE II.

368. IF the Denſity is every where as any power m of the Diſtances CN (x); then will $\dot{z}x^m = a^m\dot{x}$: and if the circular arch QN be $=y$; then will $\dot{y}^2 + \dot{x}^2 = \dot{z}^2$, and ($\dot{z}^2x^{2m} =$) $\dot{y}^2x^{2m} + \dot{x}^2x^{2m} = a^{2m}\dot{x}^2$, or $\dot{y}^2x^{2m} = \dot{x}^2 \times \overline{a^{2m} - x^{2m}}$; whence $\dot{y} = \frac{\dot{x}}{x^m}\sqrt{a^{2m} - x^{2m}}$. If the circular arch deſcribed with the radius CA$=b$, meets the line CN produced, in n, and if the arch An$=u$, then will CA (b) : CN (x) :: $\dot{u} : \dot{y} = \frac{x\dot{u}}{b}$: therefore $\dot{y} =$ $\frac{x\dot{u}}{b} = \frac{\dot{x}}{x^m}\sqrt{a^{2m} - x^{2m}}$, or $\dot{u} = \frac{b\dot{x}}{x^{m+1}}\sqrt{a^{2m} - x^{2m}}$.

In order to conſtruct the Curve, ſuppoſe $v = x^{-m}\sqrt{a^{2m} - x^{2m}}$; then will $x^{2m}vv = a^{2m} - x^{2m}$, or $x^{2m}v.v + x^{2m} = a^{2m}$; and ſo $\frac{\sqrt{1+vv}}{a^m} = x^{-m}$; whoſe fluxion will be $\frac{v\dot{v}}{ma^m\sqrt{1+vv}} =$ $-\dot{x}x^{-m-1}$: and ſince $\frac{a^{2m}}{1+vv} = x^{2m}$, ſubſtracting each ſide of this Equation from a^{2m}, then will $\frac{a^{2m}vv}{1+vv} = a^{2m} - x^{2m}$. Whence $\frac{\dot{x}b}{x^{m+1}}\sqrt{a^{2m} - x^{2m}} = -\frac{bvv\dot{v}}{m \times \overline{1+vv}} = \frac{b\dot{v}}{m \times \overline{1+vv}} - \frac{b\dot{v}}{m}$: Conſequently if D expreſſes the arch whoſe radius and tangent are as unity to $v = \left(\frac{\sqrt{a^{2m} - x^{2m}}}{x^m}\right)$, we ſhall have $\frac{b}{m}\text{D} - \frac{bv}{m} =$ u. Now ſince the angle NCQ or the arches An, and CN are given, the point N of the Curve required, will be given alſo.

THEO-

THEOREM I.

369. SUppoſe $z^{rn}v^{m}$, whoſe Fluxion is $rn\dot{z}z^{rn-1}v^{m}+mz^{rn}\dot{v}v^{m-1}$; which may be reduced to this other form, $rnv\dot{z}+mz\dot{v}$ x into $z^{rn-1}v^{m-1}$.

Now if $v=c+fz^{n}$, then will $\dot{v}=nf\dot{z}z^{n-1}$; and ſubſtituting for v and $\dot{v}$, their values into the Fluxions above, we ſhall have $rn\dot{z}z^{rn-1}v^{m}+mnf\dot{z}z^{rn+n-1}v^{m-1}$, and $ern\dot{z}+\overline{r+m},nf\dot{z}z^{n}$ x into $z^{rn-1}v^{m-1}$. Therefore, if $\dot{A}=\dot{z}z^{rn-1}v^{m}$, $\dot{B}=\dot{z}z^{rn+n-1}v^{m-1}$, $\dot{C}=\dot{z}z^{rn-1}v^{m-1}$; theſe two laſt Fluxions will become $rn\dot{A}+mnf\dot{B}$, and $ern\dot{C}+\overline{r+m},nf\dot{B}$. But ſince $z^{rn}v^{m}$ is the fluent of both theſe Fluxions; we ſhall have $rA+mfB=\frac{1}{n}z^{rn}v^{m}$, $erC+\overline{r+m},fB=\frac{1}{n}z^{rn}v^{m}$, and $rA+mfB=erC+\overline{r+m},fB$; or $A=eC+fB$. And ſubſtituting for fB its value $A-eC$, into $rA+mfB=\frac{1}{n}z^{rn}v^{m}$, it will beome $\overline{r+m}A-meC=\frac{1}{n}z^{rn}v^{m}$. Conſequently any one of the Fluents A, B, C, being given, the other two may be found.

If $v=e+fz^{n}+gz^{2n}$, then will $\dot{v}=nf\dot{z}z^{n-1}+2ng\dot{z}z^{2n-1}$; and ſubſtituting theſe values of v, $\dot{v}$, into $rn\dot{z}z^{rn-1}v^{m}+mz^{rn}\dot{v}v^{m-1}$, and $rnv\dot{z}+mz\dot{v}$ x into $z^{rn-1}v^{m-1}$, we ſhall have $rn\dot{z}z^{rn-1}v^{m}+mnf\dot{z}z^{rn+n-1}v^{m-1}+2mng\dot{z}z^{rn+2n-1}v^{m-1}$, and $ern\dot{z}+\overline{r+m},nf\dot{z}z^{n}+\overline{r+2m},ng\dot{z}z^{2n}$ x into $z^{rn-1}v^{m-1}$. If therefore $\dot{A}=\dot{z}z^{rn-1}v^{m}$, $B=\dot{z}z^{rn+n-1}v^{m-1}$, $\dot{C}=\dot{z}z^{rn+2n-1}v^{m-1}$, $\dot{D}=\dot{z}z^{rn-1}v^{m-1}$; we ſhall find in the ſame manner as above, that $rA+mfB+2mgC=\frac{1}{n}z^{rn}v^{m}$, $erD+\overline{r+m},fB+\overline{r+2m},gC=\frac{1}{n}z^{rn}v^{m}$, and $rA+mfB+2mgC=erD+\overline{r+m},fB+\overline{r+2m},gC$, or $A=eD+fB+gC$. And ſubſtituting for B, C, their values taken in the laſt Equation, into $rA+mfB+2mgC=\frac{1}{n}z^{rn}v^{m}$, we ſhall have $\overline{r+m}A-meD+mgC$

$=\frac{1}{n}z^{rn}v^{m}$, and $\overline{r+2m}\,A-mfB-2emD=\frac{1}{n}z^{rn}v^{m}$. Conſequently any two of theſe fluents A, B, C, D, being given, the reſt may be found.

THEOREM II.

370. SUppoſe $z^{rn}v^{m}u^{s}$, whoſe Fluxion is $rn\dot{z}z^{rn-1}v^{m}u^{s}+m\dot{v}v^{m-1}z^{rn}u^{s}+s\dot{u}u^{s-1}z^{rn}v^{m}$; which may be reduced to this form $rn\dot{z}z^{rn-1}vv^{m-1}u^{s}+m\dot{v}v^{m-1}z^{rn}us+s\dot{u}u^{s-1}z^{rn}v^{m}$, or to this $rn\dot{z}z^{rn-1}v^{m}uu^{s-1}+m\dot{v}v^{m-1}z^{rn}u^{s}+s\dot{u}u^{s-1}z^{rn}v^{m}$.

Now if $v=e+fz^{n}$, $u=g+hz^{n}$; then will $\dot{v}=nf\dot{z}z^{n-1}$, $\dot{u}=nh\dot{z}z^{n-1}$; and ſubſtituting theſe values of v, u, $\dot{v}$, $\dot{u}$, into the Fluxions above, we ſhall have $rn\dot{z}z^{rn-1}v^{m}u^{s}+mnf\dot{z}z^{rn+n-1}v^{m-1}u^{s}+snh\dot{z}z^{rn+n-1}v^{m}u^{s-1}$, and $ern\dot{z}z^{rn-1}v^{m-1}u^{s}+\overline{r+m},nf\dot{z}z^{rn+n-1}v^{m-1}u^{s}+snh\dot{z}z^{rn+n-1}v^{m}u^{s-1}$, and $rng\dot{z}z^{rn-1}v^{m}u^{s-1}+\overline{r+s},nh\dot{z}z^{rn+n-1}v^{m}u^{s-1}+mnf\dot{z}z^{rn+n-1}v^{m-1}u^{s}$. Therefore if $\dot{A}=\dot{z}z^{rn-1}v^{m}u^{s}$, $B=\dot{z}z^{rn+n-1}v^{m-1}u^{s}$, $\dot{C}=\dot{z}z^{rn+n-1}v^{n}u^{s-1}$, $\dot{D}=\dot{z}z^{rn-1}v^{m-1}u^{s}$, $\dot{E}=\dot{z}z^{rn-1}v^{m}u^{s-1}$, we ſhall have $rA+mfB+shC=\frac{1}{n}z^{rn}v^{m}u^{s}$, and $erD+\overline{r+m},fB+shC=\frac{1}{n}z^{rn}v^{m}u^{s}$, and $rgE+\overline{r+s},hC+mfB=\frac{1}{n}z^{rn}v^{m}u^{s}$. Now if we add the firſt Equation to the other two, we ſhall have $A=eD+fB$, and $A=gE+hC$; or $A-eD=fB$, $A-gE=hC$: and ſubſtituting the values of fB, hC, into $rA+mfB+shC=\frac{1}{n}z^{rn}v^{m}u^{s}$: we ſhall find $\overline{r+m},A-meD+shC=\frac{1}{n}z^{rn}v^{m}u^{s}$, and $\overline{r+s}\,A+mfB-sgE=\frac{1}{n}z^{rn}v^{m}u^{s}$. Conſequently any two of the Fluents A, B, C, &c. being given the reſt may be found.

EX-

EXAMPLE I.

371. LET C expreſs the fluent of $\dot{x}\sqrt{aa-xx}$, and let it be required to find the fluent of $x^2\dot{x}\sqrt{aa-xx}$.

Suppoſe $\sqrt{aa-xx}=v^{m-1}$, $r=\frac{1}{2}$, $m-1=\frac{1}{2}$, or $m=\frac{3}{2}$, $e=aa$; $f=-1$, $z^n=xx$, and $n=2$: then will * $\dot{C}\ (=\dot{z}z^{rn-1}v^{m-1})$ $=\dot{x}\sqrt{aa-xx}$, and $\dot{B}\ (=\dot{z}z^{rn+n-1}v^{m-1})=x^2\dot{x}\sqrt{aa-xx}$: Therefore ſubſtituting the values of e, r, n, m, z, v, into erC, $+\overline{r+m}, fB=\frac{1}{n}z^{rn}v^m$, we ſhall find $\frac{1}{2}aaC-2B=\frac{1}{2}x$ $\times\overline{aa-xx}|^{\frac{1}{2}}$, or $\frac{1}{4}aaC-\frac{1}{4}x\times\overline{aa-xx}|^{\frac{3}{2}}=B=$ to the Fluent required. * Art. 369.

EXAMPLE II.

372. LET A expreſs the Fluent of $\dot{x}\sqrt{aa-xx}$, and let it be required to find the Fluent of $\dot{x}\times\overline{aa-xx}|^{-\frac{1}{2}}$.

Suppoſe $\sqrt{aa-xx}=v^m$, $m=\frac{1}{2}$, $aa=e$, $-f=1$, $r=\frac{1}{2}$, $n=2$; then will $\dot{A}\ (=\dot{z}z^{rn-1}v^m)=\dot{x}\sqrt{aa-xx}$, and $\dot{C}\ (=\dot{z}z^{rn-1}v^{m-1})$ $=\dot{x}\times\overline{aa-xx}|^{-\frac{1}{2}}$; therefore, ſubſtituting the values of r, m, e, n, z, v, into * $\overline{r+m}A-meC=\frac{1}{n}z^{rn}v^m$, we ſhall have * Art. 369.
$A-\frac{1}{2}aaC=\frac{1}{2}x\sqrt{aa-xx}$; or $C=\frac{2}{aa}A-\frac{x}{aa}\sqrt{aa-xx}=$ to the fluent required.

COROLLARY.

373. IF $v=e+fz^n$, and s, t, expreſs any whole numbers whatſoever; then if the fluent of $\dot{z}z^{rn-1}v^m$ be given, the fluent of $\dot{z}z^{rn\pm sn-1}v^m$, and of $\dot{z}z^{rn-1}v^{m\pm t}$ may be found.

Likewiſe, if $v=e+fz^{n}+gz^{2n}$, and the Fluent of $\dot{z}z^{rn-1}v^{m-1}$, $\dot{z}z^{rn+n-1}v^{m-1}$ are given: then if s, t expreſs any whole numbers; the fluent of $\dot{z}z^{rn\pm sn-1}v^{m-1}$, and of $\dot{z}z^{rn-1}v^{m\pm t}$, may be found; and ſo on *ad infinitum*.

N. B. The following Tables of Fluents have been conſtructed by the help of the forms exhibited in the firſt *Theorem*, which may be continued *ad infinitum*, ſo as to contain not only the Fluents of Binomial expreſſions but likewiſe thoſe of Trinomials, Quadrinomials, *&c.* multiplied or divided by others which are rational or irrational. But as the ſeveral Caſes of each form are found by the help of one or ſometimes two given ones, we ſhall ſhew in the following Examples how they are to be found, as alſo how to proceed in any other Caſe.

EXAMPLE I.

374. TO *find the* Fluent *of* $\frac{d\dot{z}z^{\frac{1}{2}n-1}}{e+fz^{n}}$; *which is the Caſe of Form* II. *when* $r=0$.

If D expreſſes the circular arch in degrees, whoſe radius and tangent are as r $(=\sqrt{\frac{e}{f}})$ to $z^{\frac{1}{2}n}$ when $\frac{e}{f}$ is poſitive, or if L expreſſes the Logarithm of $\frac{z^{\frac{1}{2}n}+r}{\sqrt{z^{n}-rr}}$ when $\frac{e}{f}$ is negative; then will $\frac{2d}{ne}r$DK be the Fluent required in the firſt Caſe, and $\frac{2d}{ne}r$LR in the ſecond, ſuppoſing K $=0.017453292519943$ *&c.* and R $=2.302585092994$ *&c.*

For the Fluxion of the arch whoſe radius and tangent are, as r to $z^{\frac{1}{2}n}$, will be * $\frac{\frac{1}{2}nrr\dot{z}z^{\frac{1}{2}n-1}}{rr+z^{n}}=\frac{\frac{1}{2}ne\dot{z}z^{\frac{1}{2}n-1}}{e+fz^{n}}$: But this is likewiſe the Fluxion of * rDK: Therefore that Fluxion multiplied by $\frac{2d}{ne}$ will give $\frac{d\dot{z}z^{\frac{1}{2}n-1}}{e+fz^{n}}$ for the Fluxion required. And the

* Art. 247.

* Art. 249.

the Fluxion of the Logarithm of $z^{\frac{1}{2}n}+r$, will be* $\frac{\frac{1}{2}n\dot{z}z^{\frac{1}{2}n-1}}{z^{\frac{1}{2}n}+r}$, or by multiplying and dividing that Fluxion by $z^{\frac{1}{2}n}-r$, $=\frac{\frac{1}{2}n\dot{z}z^{n-1}-\frac{1}{2}nr\dot{z}z^{n-1}}{z^{n}-rr}$: and the Fluxion of the Logarithm of $\sqrt{z^{n}-rr}$, will be $\frac{\frac{1}{2}n\dot{z}z^{n-1}}{z^{n}-rr}$; which being ſubſtracted from the former, will give $\frac{-\frac{1}{2}nr\dot{z}z^{\frac{1}{2}n-1}}{z^{n}-rr}$ for the Fluxion of LR. Therefore this Fluxion multiplied by $\frac{2d}{ne}r$, will give $\frac{d\dot{z}z^{\frac{1}{2}n-1}}{e+fz^{n}}$ for the fluxion required; becauſe $-rr=\frac{e}{f}$.

* *Art.* 246

EXAMPLE II.

375. TO *find the* Fluent *of* $\frac{d\dot{z}z^{-\frac{1}{2}n-1}}{e+fz^{n}}$, *which is the caſe of* Form II, *when* $r=-1$.

If D expreſſes the circular arch in degrees, whoſe radius and tangent are as $r(=\sqrt{\frac{f}{e}})$ to $z^{-\frac{1}{2}n}$ when $\frac{f}{e}$ is poſitive, or if L expreſſes the Logarithm of $\frac{z^{-\frac{1}{2}n}+r}{\sqrt{z^{-n}-rr}}$ when $\frac{f}{e}$ is negative; then will $\frac{-2d}{nez^{\frac{1}{2}n}}+\frac{2d}{ne}r$DK be the fluent in the firſt Caſe, and $\frac{-2d}{nez^{\frac{1}{2}n}}+\frac{2d}{ne}r$LR in the ſecond.

For the Fluxion of the arch whoſe radius and tangent are as r to $z^{-\frac{1}{2}n}$, will be $\frac{-\frac{1}{2}nrr\dot{z}z^{-\frac{1}{2}n-1}}{rr+z^{-n}}=-\frac{\frac{1}{2}nf\dot{z}z^{-\frac{1}{2}n-1}}{f+ez^{-n}}$, and the Fluxion of $\frac{-2d}{nez^{\frac{1}{2}n}}$, is $\frac{\dot{z}d}{ez^{\frac{1}{2}n+1}}$; therefore $\frac{\dot{z}d}{ez^{\frac{1}{2}n+1}}-\frac{\frac{1}{2}nf\dot{z}z^{-\frac{1}{2}n-1}}{f+ez^{-n}}$

$\times \frac{2d}{ne} = \frac{d\dot{z}z^{-\frac{1}{2}n-1}}{f+ez^{-n}} = \frac{d\dot{z}z^{-\frac{1}{2}n-1}}{e+fz^{n}}$ will be the Fluxion required. And the Fluxion of the Logarithm of $z^{-\frac{1}{2}n}+r$, will be $\frac{-\frac{1}{2}n\dot{z}z^{-\frac{1}{2}n-1}}{z^{-\frac{1}{2}n}+r}$, or multiplying both the numerator and the denominator by $z^{-\frac{1}{2}n}-r$, it will be $=\frac{-\frac{1}{2}n\dot{z}z^{-n-1}+\frac{1}{2}nr\dot{z}z^{-\frac{1}{2}n-1}}{z^{-n}-rr}$; and the Fluxion of the Logarithm of $\sqrt{z^{-n}-rr}$ will be $\frac{-\frac{1}{2}n\dot{z}z^{-\frac{1}{2}n-1}}{z^{-n}-rr}$; which being ſubſtracted from the firſt, will give $\frac{\frac{1}{2}nr\dot{z}z^{-\frac{1}{2}n-1}}{z^{-n}-rr}$ for the Fluxion of LR, and this being multiplied by $\frac{2d}{ne}r$, and added to the Fluxion $\frac{\dot{z}d}{ez^{\frac{1}{2}n+1}}$ of $\frac{-2d}{nez^{\frac{1}{2}n}}$, will give $\frac{\dot{z}d}{ez^{\frac{1}{2}n+1}} - \frac{df\dot{z}z^{-\frac{1}{2}n-1}}{\overline{ez^{-n}+f}\times e} = \frac{d\dot{z}z^{-\frac{1}{2}n-1}}{e+fz^{n}}$ for the Fluxion required.

EXAMPLE III.

376. TO *find the* Fluent *of* $\frac{d\dot{z}z^{-1}}{\sqrt{e+fz^{n}}}$, *which is the Caſe of Form* V. *when* $r=0$.

If D expreſſes the arch in degrees whoſe radius and tangent are as r $(=\sqrt{e})$ to $\sqrt{e+fz^{n}}$ when e is negative, or if L expreſſes the Logarithm of $\frac{r+\sqrt{e+fz^{n}}}{\sqrt{fz^{n}}}$ when e is poſitive; then will $\frac{-2d}{ne}r\text{DK}$ be the Fluent in the firſt Caſe, and $\frac{-2d}{ne}r\text{LR}$ in the ſecond. For the Fluxion of the arch whoſe ſine and tangent are as r to $\sqrt{e+fz^{n}}$, will be $\frac{-\frac{1}{2}ne\dot{z}z^{-1}}{\sqrt{e+fz^{n}}}$, becauſe $-rr=e$. Therefore, this fluent multiplied by $-\frac{2d}{ne}$ will give $\frac{d\dot{z}z^{-1}}{\sqrt{e+fz^{n}}}$. And the

Fluxion of the Logarithm of $r+\sqrt{e+fz^n}$, will be $\frac{1}{\sqrt{e+fz^n}}\times\frac{\frac{1}{2}nf\dot{z}z^{n-1}}{r+\sqrt{e+fz^n}}$, and multiplying both the numerator and the denominator by $-r+\sqrt{e+fz^n}$, it will be $\frac{-\frac{1}{2}nr\dot{z}z^{\frac{1}{2}n-1}}{\sqrt{e+fz^n}}+\frac{1}{2}n\dot{z}z^{-1}$; and the Fluxion of the Logarithm of $\sqrt{fz^n}$, will be $\frac{1}{2}n\dot{z}z^{-1}$; which being ſubſtracted from the firſt, and the remaining part multiplied by $-\frac{2d}{ne}r$, will give $\frac{d\dot{z}z^{\frac{1}{2}n-1}}{\sqrt{e+fz^n}}$ for the Fluxion required.

PROBLEM XXVI.

377. *To reduce any* Fraction *ſuch as* $\frac{1}{e+fz+gzz+hz^3+\&c.}$ *into as many ſingle ones, as there are roots in the Denominator.*

Suppoſe $e+fz+gzz+hz^3+\&c.=0$, and $d-z$ to be one of the roots, and $\frac{1}{e+fz+gzz+hz^3+\&c.}=\frac{r+sz+tzz+\&c.}{a+bz+czz+\&c.}+\frac{v}{d-z}$; that is, that the Fractions are found already and reduced under the ſame denomination, except the one $d-z$: The letters r, s, t, u, expreſs indeterminate but conſtant quantities.

Now this laſt Fraction $\frac{r+sz+tzz+\&c.}{a+bz+czz+\&c.}+\frac{v}{d-z}$, being reduced under the ſame denomination, will give

$$\begin{matrix} dr+dsz+dtzz & \\ -rz-szz-tz^3 & \\ au+buz+cuzz+uz^3 & \end{matrix}$$

for the numerator; which ought to be equal to the numerator (1) of the Fraction propoſed. Therefore, if $dr+au=1$, and $dsz-rz+buz=0$, $dtzz-szz+cuzz=0$,

$uz^3-tz^3=0$, or $ds+bu=r$, $dt+cu=s$, $u=t$; then will $r=ddt+dcu+bu$, becauſe $s=dt+cu$, and ($dr+au=$) $d^3u+ddcu+dbu+au=1$, becauſe $t=u$, or $u=\overline{d^3+ddc+db+a}^{\,-1}$: and conſequently $\frac{u}{d-z}=\frac{\overline{d^3+ddc+db+a}^{\,-1}}{d-z}$. But it is evident, that what has been ſaid with regard to the root $d-z$, may be likewiſe applied to any other. Therefore, &c.

PROBLEM XXVII.

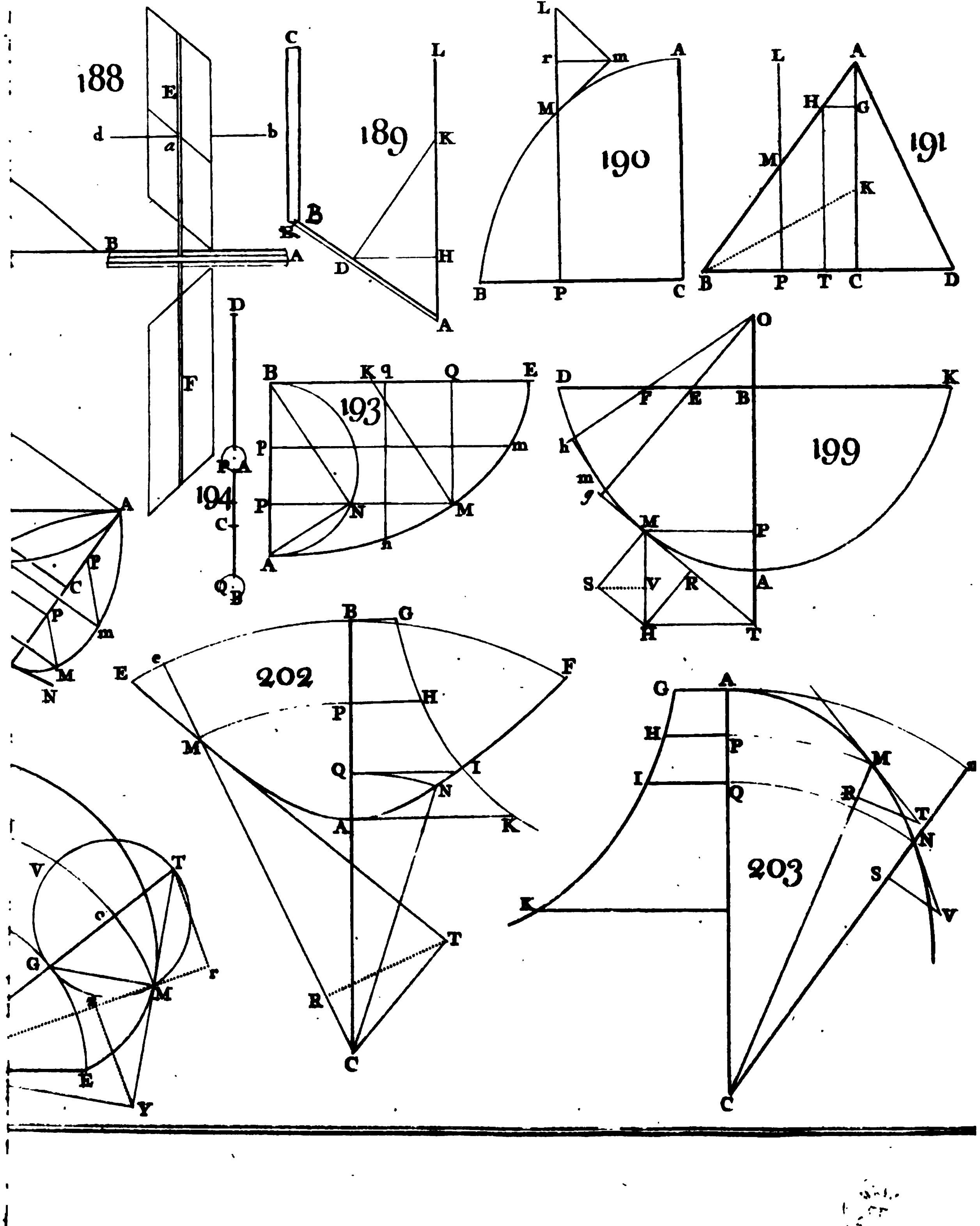
378. *FROM the given denominator* $e+fz+gzz+hz^3+$ &c: *or its equal* $a+bz+czz+$ &c. × *into* $d-z$; *to find the* Numerator $\overline{d^3+ddc+db+a}^{\,-1}$ *of the Fraction whoſe denominator is any root* $d-z$ *of the given denominator.*

Divide the Fluxion of the Denominator $a+bz+czz+$ &c: by $\dot{z}$, and ſubſtitute for z its root d, then by changing all the ſigns, you will have the Numerator required.

For the Fluxion of the ſaid Denominator will be $-\dot{z}$ × into $a+bz+czz+$, &c. $+b\dot{z}+2c\dot{z}z+$ &c:, which being divided by $\dot{z}$, and d being ſubſtituted for z, will give $-a-bd-cdd-d^3+$, &c.

N. B. The root which is ſubſtituted for z, muſt be always preceded by its proper Sign; that is, by + when poſitive, and by − when negative.

FINIS.


188
E
d
a
b
B
A
F
C
B
E
189
L
K
D
H
A
L
r
m
A
M
190
B
P
C
L
A
H
G
M
191
K
B
P
T
C
D
D
P
A
194
C
Q
B
B
K
q
Q
E
193
P
m
P
N
M
h
A
O
D
F
E
B
K
h
199
m
g
M
P
S
V
R
A
H
T
A
P
C
P
m
M
N
B
G
e
E
202
F
P
H
M
Q
I
N
A
K
T
R
C
G
A
H
P
I
Q
M
R
T
N
203
S
V
K
C
V
T
c
G
M
r
E
Y

Explanation of the following TABLES.

IN the firſt vertical line, are placed the different values 3, 2, 1, 0, —1,—2, *&c*: of the quantity r, and over againſt them, the fluents of the caſes of the general Form, wherein r expreſſes thoſe numbers. For Example, when $r=0$, the firſt Form $\left(\frac{d\dot{z}z^{rn-1}}{e+fz^{n}}\right)$ becomes $\frac{d\dot{z}z^{-1}}{e+fz^{n}}$; whoſe fluent $\frac{-d}{nf}\mathrm{lR}$ ſtands over-againſt 0 : and ſo on. The letter D at the bottom, expreſſes the circular arch in degrees, whoſe radius and tangent are as r to T ; L expreſſes the
logarithm of the quantity to which it is ſuppoſed equal ; K the * * *Art.* 248.
reciprocal trigonometrical *Modulus* ; R the * reciprocal logarith- * *Art.* 243.
mical *Modulus* ; and O is * = r D K when r is the root of a ne- * *Art.* 249.
gative quantity (which in practice is taken poſitive) or * O =rLR * *Art.* 244.
when r is the root of a poſitive quantity.

The Fluents of the different Caſes of the IVth and VIth Forms, are found by multiplying the reſpective fluents of the IIId and Vth Forms by the correſpondent quantities after the Sign ×, and by adding the next following one with its Signs: For example when $r=0$, the IVth Form becomes $\frac{d\dot{z}z^{-1}}{\sqrt{e+fz^{n}}}$; whoſe fluent $\overline{\frac{1}{n}dP-\frac{1}{n}dO}\times\frac{1}{e}-\frac{1}{ne}dP=-\frac{2}{ne}dO$, ſtands over-againſt 0 : when $r=1$, that Form becomes $\frac{d\dot{z}z^{n-1}}{\sqrt{e+fz^{n}}}$; whoſe fluent $\frac{\overline{2e+2fz^{n}}}{3nf}dP\times\frac{3}{e}-\frac{2z^{n}}{ne}dP=\frac{2}{nf}dP$, ſtands over-againſt 1 : And ſo on in any other Caſe.

The values of O, o at the bottom under the VIIth and VIIIth Forms are common to both.

r.	Form I. $\frac{dz z^{rn-1}}{e+fz^n}$.
5.	$\frac{dz^{4n}}{4nf}-\frac{dez^{3n}}{3nff}+\frac{deez^{2n}}{2nf^3}-\frac{de^3z^n}{nf^4}+\frac{de^4}{nf^5}$ LR.
4.	$\frac{dz^{3n}}{3nf}-\frac{dez^{2n}}{2nff}+\frac{deez^n}{nf^3}-\frac{de^3}{nf^4}$ LR.
3.	$\frac{dz^{2n}}{2nf}-\frac{dez^n}{nff}+\frac{dee}{nf^3}$ LR.
2.	$\frac{dz^n}{nf}-\frac{de}{nff}$ LR.
1.	$\frac{d}{nf}$ LR.
0.	$\frac{-d}{ne}$ lR.
−1.	$\frac{-d}{nez^n}+\frac{df}{nee}$ lR.
−2.	$\frac{-d}{2nez^{2n}}+\frac{df}{neez^n}-\frac{dff}{ne^3}$ lR.
−3.	$\frac{-d}{3nez^{3n}}+\frac{df}{2neez^{2n}}-\frac{dff}{ne^3z^n}+\frac{df^3}{ne^4}$ lR.
−4.	$\frac{-d}{4nez^{4n}}+\frac{df}{3neez^{3n}}-\frac{dff}{2ne^3z^{2n}}+\frac{df^3}{ne^4z^n}-\frac{df^4}{ne^5}$ lR.

L $=$ log. $\frac{e+fz^n}{e}$, l $=$ log. $\frac{e+fz^n}{z^n}$. R $=2.302585092994$ &c.

r.	FORM II. $\frac{d\dot{z}z^{rn+\frac{1}{2}n-1}}{e+fz^n}$.
4.	$\frac{2dz^{\frac{1}{2}n}}{7nf}-\frac{2dez^{\frac{1}{2}n}}{5nff}+\frac{2deez^{\frac{1}{2}n}}{3nf^3}-\frac{2de^3z^{\frac{1}{2}n}}{nf^4}-\frac{2e^3}{nf^4}dO$
3.	$\frac{2dz^{\frac{1}{2}n}}{5nf}-\frac{2dez^{\frac{1}{2}n}}{3nff}+\frac{2deez^{\frac{1}{2}n}}{nf^3}-\frac{2ee}{nf^3}dO$.
2.	$\frac{2dz^{\frac{1}{2}n}}{3nf}-\frac{2dez^{\frac{1}{2}n}}{nff}+\frac{2e}{nff}dO$.
1.	$\frac{2dz^{\frac{1}{2}n}}{nf}-\frac{2}{nf}dO$.
0.	$\frac{2}{ne}dO$.
−1.	$\frac{-2d}{nez^{\frac{1}{2}n}}+\frac{2}{ne}do$.
−2.	$\frac{-2d}{3nez^{\frac{1}{2}n}}+\frac{2df}{neez^{\frac{1}{2}n}}-\frac{2f}{nee}do$.
−3.	$\frac{-2d}{5nez^{\frac{1}{2}n}}+\frac{2df}{3neez^{\frac{1}{2}n}}-\frac{2dff}{ne^3z^{\frac{1}{2}n}}+\frac{2ff}{ne^3}do$.
−4.	$\frac{-2d}{7nez^{\frac{1}{2}n}}+\frac{2df}{5neez^{\frac{1}{2}n}}-\frac{2dff}{3ne^3z^{\frac{1}{2}n}}+\frac{2df^3}{ne^4z^{\frac{1}{2}n}}-\frac{2f^3}{ne^4}do$.

$r=\sqrt{\frac{-e}{f}}$, $T=z^{\frac{1}{2}n}$, $S=\sqrt{\frac{e+fz^n}{f}}$ $rDK=O$, or $L=\frac{r+T}{S}$, $\{rLR=O$.

$r=\sqrt{\frac{-f}{e}}$, $t=\frac{1}{z^{\frac{1}{2}n}}$, $s=\sqrt{\frac{e+fz^n}{ez^n}}$ $rDK=o$, or $l=\frac{r+t}{s}$, $rlR=o$.

r.	FORM III. $d\dot{z}z^{rn-1}\sqrt{e+fz^n}$.	IV. $\frac{d\dot{z}z^{rn-1}}{\sqrt{e+fz^n}}$.
4.	$\frac{-96e^4+48e^3fz^n-36eeffz^{2n}+30ef^3z^{3n}+210f^4z^{4n}}{945nf^4}dP$.	$(\times\frac{9}{e}-\frac{2z^{4n}}{ne}dP$.
3.	$\frac{16e^3-8eefz^n+6effz^{2n}+30f^3z^{3n}}{105nf^3}dP$.	$\times\frac{7}{e}-\frac{2z^{3n}}{ne}dP$.
2.	$\frac{-4ee+2efz^n+6ffz^{2n}}{15nff}dP$.	$\times\frac{5}{e}-\frac{2z^{2n}}{ne}dP$.
1.	$\frac{2e+2fz^n}{3nf}dP$.	$\times\frac{3}{e}-\frac{2z^n}{ne}dP$.
0.	$\frac{2}{n}dP-\frac{2}{n}dO$.	$\times\frac{1}{e}-\frac{2}{ne}dP$.
−1.	$\frac{-1}{nz^n}dP-\frac{f}{ne}dO$.	$\times\frac{-1}{e}-\frac{2}{nez^n}dP$.
−2.	$\frac{-2e-fz^n}{4nez^{2n}}dP+\frac{ff}{4nee}dO$.	$\times\frac{-3}{e}-\frac{2}{nez^{2n}}dP$.
−3.	$\frac{-8ee-2efz^n+3ffz^{2n}}{24neez^{3n}}dP-\frac{f^3}{8ne^3}dO$.	$\times\frac{-5}{e}-\frac{2}{nez^{3n}}dP$.
−4.	$\frac{-48e^3-8eefz^n+10effz^{2n}-15f^3z^{3n}}{192e^3z^{4n}}dP+\frac{5f^4}{64ne^4}dO$.	$(\times\frac{-7}{e}-\frac{2}{nez^{4n}}dP$.

$r=\sqrt{e}$, $P=T=\sqrt{e+fz^n}$, $S=\sqrt{e+fz^n}$, $rDK=O$, or $L=\frac{r+T}{S}$, $rLR=O$.

r.	Form V. $d\dot{z}z^{rn+\frac{1}{2}n-1}\sqrt{e+fz^n}$.	VI. $\frac{d\dot{z}z^{rn+\frac{1}{2}n-1}}{\sqrt{e+fz^n}}$.
4.	$\frac{-105e^4z^n+70e^3fz^{2n}-56eeffz^{3n}+48f^3ez^{4n}+384f^4z^{5n}}{1920nf^4}dP$. $\left(+\frac{35e^5}{640nf^5}dO\right.$.	$\left(\times\frac{10}{e}-\frac{2z^{5n}}{ne}dP\right.$.
3.	$\frac{15e^3z^n-10eefz^{2n}+8effz^{3n}+48f^3z^{4n}}{192nf^3}dP-\frac{5e^4}{64nf^4}dO$.	$\left(\times\frac{8}{e}-\frac{2z^{4n}}{ne}dP\right.$.
2.	$\frac{-3eez^n+2efz^{2n}+8ffz^{3n}}{24nff}dP+\frac{e^3}{8nf^3}dO$.	$\times\frac{6}{e}-\frac{2z^{3n}}{ne}dP$
1.	$\frac{ez^n+2fz^{2n}}{4nf}dP-\frac{ee}{4nff}dO$.	$\times\frac{4}{e}-\frac{2z^{2n}}{ne}dP$.
0.	$\frac{z^n}{n}dP+\frac{e}{nf}dO$.	$\times\frac{2}{e}-\frac{2z^n}{ne}dP$.
—1.	$\frac{-2}{n}dP+\frac{2}{n}dO$.	$\times\frac{0}{e}-\frac{2}{ne}dP$.
—2.	$\frac{-2e-2fz^n}{3nez^n}dP$.	$\times\frac{-2}{e}-\frac{2}{nez^n}dP$.
—3.	$\frac{-6ee-2efz^n+4ffz^{2n}}{15neez^{2n}}dP$.	$\times\frac{-4}{e}-\frac{2}{nez^{2n}}dP$.
—4.	$\frac{-30e^3-6eefz^n+8effz^{2n}-16f^3z^{3n}}{105ne^3z^{3n}}dP$.	$\left(\times\frac{-6}{e}-\frac{2}{nez^{3n}}dP\right.$.

$$r=\sqrt{f},\ P=T=\sqrt{\frac{e+fz^n}{z^n}},\ S=\sqrt{\frac{e}{z^n}},\ rDK=O,\ \text{or}\ L=\frac{r+T}{S},\ (rLR=O.$$

r.	FORM VII. $d\dot{z}z^{rn-1}\sqrt{e+fz^n+gz^{2n}}$.
3.	$\left(+\frac{\begin{array}{l}-16eegg\\+24effg-5f^4\end{array}}{128ng^4}dO.\right.$ $\frac{\begin{array}{l}+15f^3+24eggz^n\\-52efg-10ffgz^n+8fggz^{2n}+48g^3z^{3n}\end{array}}{192ng^3}dP.$
2.	$\frac{\begin{array}{l}+8cg\\-3ff+2fgz^n+8ggz^{2n}\end{array}}{24ngg}dP+\frac{f^3-4efg}{16ng^3}dO.$
1.	$\frac{f+2gz^n}{4ng}dP+\frac{4eg-ff}{8ngg}dO.$
0.	$\frac{1}{n}dP+\frac{f}{2ng}dO-\frac{1}{n}do.$
−1.	$\frac{-1}{nz^n}dP-\frac{f}{2ne}do+\frac{1}{n}dO.$
−2.	$\frac{-2e-fz^n}{4nez^{2n}}dP+\frac{ff-4eg}{8nee}do.$
−3.	$\frac{\begin{array}{l}-8egz^{2n}\\-8ee-2efz^n+3ffz^{2n}\end{array}}{24neez^{3n}}dP+\frac{4efg-f^3}{16ne^3}do.$
−4.	$\left(+\frac{\begin{array}{l}-24effg\\+16eegg+5f^4\end{array}}{128ne^4}do.\right.$ $\frac{\begin{array}{l}-24eegz^{2n}-15f^3z^{3n}\\-48e^3-8eefz^n+10effz^{2n}+52efgz^{3n}\end{array}}{192ne^3z^{4n}}dP$

$$P=\sqrt{e+fz^n+gz^{2n}},\ \mathrm{r}=\sqrt{g},\ T=\frac{\frac{1}{2}f+gz^n}{\sqrt{e+fz^n+gz^{2n}}}$$

$$S=\sqrt{\frac{\frac{1}{4}ff-eg}{e+fz^n+gz^{2n}}},\ \mathrm{r}DK=O,\ \text{or}\ L=\frac{\mathrm{r}+T}{S},\ \mathrm{r}LR=O.$$

r.	FORM VIII. $\dfrac{d\dot{z}z^{rn-1}}{\sqrt{e+fz^n+gz^{2n}}}$:
4.	$\dfrac{\begin{array}{l}+15ff\\ -16eg-10fgz^n+8ggz^{2n}\end{array}}{24ng^3}d\mathrm{P}+\dfrac{12efg-5f^3}{16ng^4}d\mathrm{O}.$
3.	$\dfrac{-3f+2gz^n}{4ngg}d\mathrm{P}+\dfrac{3ff-4eg}{8ng^3}d\mathrm{O}.$
2.	$\dfrac{1}{ng}d\mathrm{P}-\dfrac{f}{2ngg}d\mathrm{O}.$
1.	$\dfrac{1}{ng}d\mathrm{O}.$
0.	$\dfrac{-1}{ne}d\mathrm{o}.$
—1.	$\dfrac{-1}{nez^n}d\mathrm{P}+\dfrac{f}{2nee}d\mathrm{o}.$
—2.	$\dfrac{-2e+3fz^n}{4neez^{2n}}d\mathrm{P}+\dfrac{4eg-3ff}{8ne^3}d\mathrm{o}.$
—3.	$\dfrac{\begin{array}{l}-15ffz^{2n}\\ -8ee+10fz^n+16egz^{2n}\end{array}}{24ne^3z^{3n}}d\mathrm{P}+\dfrac{5f^3-12efg}{16ne^4}d\mathrm{o}.$

$r=\sqrt{e}$, $\mathrm{t}=\dfrac{e+\frac{1}{2}fz^n}{\sqrt{e+fz^n+gz^{2n}}}$, $\mathrm{S}=z^n\sqrt{\dfrac{\frac{1}{4}ff-eg}{e+fz^n+gz^{2n}}}$,

$r\mathrm{DK}=\mathrm{o}$, or $\mathrm{L}=\dfrac{r+\mathrm{t}}{\mathrm{S}}$, $r\mathrm{LR}=\mathrm{o}$.

E R R A T A.

PAGE 4. l. 3. $\overline{\mp a \mp u}$, r. $\overline{\pm a \mp u}$. p. 7. l. 20. FM, r. $\overline{\mathrm{FM}}^2$. p. 9. l. 5. *reduced*, r. *deduced*. p. 15. l. 24. $\frac{br \pm bs}{\sqrt{rs}}$, r. $\frac{br \pm bs}{2\sqrt{rs}}$. p. 16. l. 7. $\mathrm{MK} = \frac{b}{p}\sqrt{rs}$, r. $\mathrm{MK} = \frac{b}{a}\sqrt{rs}$. p. 18. l. 10. *after parallel to* Mm, *add and biſecting* LM. l. 14. $\overline{\mathrm{AC}}^2 a$, r. $\overline{\mathrm{AC}}^2$. p. 19 l. 3. MR, r. MQ. l. 13. $u - x$, r. $v - x$. p. 23. l. 16. $\mp$CL, r. $\mp\overline{\mathrm{CL}}^2$. p. 26. l. 17. $\mp aa \pm uu$. r. $\pm aa \mp uu$. *ibid.* $\mp aa \pm au$, r. $\pm aa \mp au$. p. 33. l. 23. *be*, r. *the*. p. 36. *fig.* 88. r. *fig.* 58. p. 42. l. 17. *upon*, r. *to*. p. 47. l. 1. MK, r. BK. p. 48. l. 5. $\overline{\mathrm{CL}}^3$ r. $\overline{\mathrm{CL}}^2$. p. 50. l. 14, *upon*, r. *to*. p. 53. l. 25. *at*. r. *on*. l. 30. EK, r. FK. p. 54. l. 27. $apy - rry$, r. $apy + rry$. p. 56. l. 8. *at*, r. *on*. p. 59. l. 10. DB, r. CB. l. 12. $\sqrt{\mathrm{DEB}} : \sqrt{\mathrm{BHD}}$, r. $\sqrt{\mathrm{CEB}} : \sqrt{\mathrm{BHC}}$. p. 68. l. 16. KPN, r. KPn *Add* 1 *to each Number of the Theorems, from page* 33 *thro' the whole firſt Book; as likewiſe to each Number of the Prob. from page* 39. p. 71. l. 8. $\overline{a+u}^n$ r. $\overline{a+x}^n$. p. 72. l. 6. $\frac{1}{a^m}$, r. $\frac{1}{a^n}$. p. 73. *fig.* 199. r. *fig.* 119. p. 74. l. 17. *del.* *a*. p. 79. l. 6. $m\dot{y}y^{m-}$, r. $m\dot{y}y^{m-1}$. p. 89. l. 5. $\ddot{v}z - \dot{v}\dot{z}$, r. $\ddot{v}z + \dot{v}\dot{z}$. p. 90. l. 6. $\dot{y}v - \dot{y}\ddot{v}$, r. $\dot{y}\dot{v} - y\ddot{v}$. l. 7. (v) r. $(\dot{v})$. p. 92. l. 8. $u \mp \frac{p\dot{u}}{a}$, r. $\dot{u} \mp \frac{p\dot{u}}{a}$. p. 102. l. 22. $x\dot{x} \times \overline{aa - xx}^{\frac{1}{2}}$, r. $x\dot{x} \times \overline{aa - xx}^{-\frac{1}{2}}$. p. 104. l. 3. $fz^n = \frac{e}{v-1}$, r. $fz^n = \frac{fe}{v-f}$. p. 106. l. *penult.* $\mathrm{B}\,\frac{m-1}{2}$, $\mathrm{C} = \frac{m-2}{3}$, r. $\mathrm{B} = \frac{m-1}{2e}$, $\mathrm{C} = \frac{m-2}{3e}$. p. 118. l. 9. $r > s$, r. $r > n$. l. 11. $r < s$, r. $r < n$. l. 18. $\frac{s+1}{1}$, r. $\frac{s+1}{s}$. p. 120. l. 4. $+\frac{1}{4}x^4$, r. $-\frac{1}{4}x^4$. l. 10. $\frac{1}{9}x$, r. $\frac{1}{9}x^9$. p. 128. l. 21. TM r. TP. p. 134. l. 20. $\sqrt{1+duu}$, r. $\sqrt{1+dduu}$. p. 141. *ant-penult. after or, add* $\frac{1-m}{1-2m}x$. p. 142. *dele the ten firſt lines.* p. 144. *after line* 13, *add the.* p. 149. l. 11 *Iſoronous*, r. *Iſocronous*. p. 152. l. 6. $\frac{2am}{1+2m}y^{m+1}$, r. $\frac{2am}{1+2m}y^{2m+1}$. p. 156

l. 12, r. $+\times\frac{rrxx}{20aa}$, r. $+\frac{rrxx}{20aa}$. p. 159. l. 25, $\frac{bn-aa}{aa}$, r. $\frac{bn-aa}{a}$. p. 161. l. 10, *dele a before Motion.* p. 162. l. 22, $\overline{CN}^2$, r. $\overline{DN}^2$. p. 176. l. 6. *the of*, r. *of*. p. 181. l. 18. $\ddot{z}$, r. $\dot{z}^2$. p. 182. l. 14. *after into the, add* Log. p. 189. l. 6. *Art.* 394, r. *Art.* 294. p. 192. l. 21. *put* M *before multiplied, and dele* M *after by.* p. 193. l. 8, cy^2, r. $c\dot{y}^2$. p. 194. l. 15, *dele* : : . p. 197. l. 11. $\dot{y}^2+\dot{x}$, r. $\dot{y}^2+\dot{x}^2$. p. 200. l. 1. x^m, r. x^{m+1}. p. 201. l. 22. *of*, r. *in*. p. 203. l. *ant-penult* $\frac{a+b}{b}$, r. $\frac{a+2b}{b}$. p. 204. l. 6, *fluxion* An, r. *fluxion of* An. p. 213. l. 1. $rn\dot{z}^{rn-1}v^m$, r. $rn\dot{z}z^{rn-1}v^m$. l. 19. z^{n-}, r. z^{n-1}. l. 22. B, r. $\dot{B}$. p. 214. l. 6. u^{-1}, r. u^{r-1}. l. 16. B, r. $\dot{B}$. p. 215. l. 13, $\overline{aa-xx}$, r. $\overline{aa-xx}^{\frac{1}{2}}$. p. 217. l. 3, $-\frac{1}{2}nr\dot{z}z^{n-1}$, r. $-\frac{1}{2}nr\dot{z}z^{\frac{1}{2}n-1}$. l. *penult.* $\frac{1}{2}fzz^{-\frac{1}{2}n-1}$, r. $\frac{1}{2}f\dot{z}z^{-\frac{1}{2}n-1}$. p. 218. l. 4. $-\frac{1}{2}n\dot{z}z^{-n+1}$, r. $-\frac{1}{2}n\dot{z}z^{-n-1}$. l. 6. $-\frac{1}{2}n\dot{z}z^{-\frac{1}{2}n-1}$, r. $-\frac{1}{2}n\dot{z}z^{-n-1}$.